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On the Steady State Motions Control Problem for Mechanical Systems with Relay Controllers

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***Abstract*—The paper presents the solution to the stabilization problem of steady state motions for a holonomic mechanical system by using relay controllers. This solution is achieved by proving new theorems on the asymptotic stability of the solution to a differential equation with a discontinuous right-hand side. The novelty of the theorems is based on the limiting inclusions construction and the use of semi-definite Lyapunov functions. As an example, the stabilization problem of steady-state motion for a five-link robot manipulator is solved by using relay controller.**

***Index Terms*—differential equation with discontinuous right-hand side, stability, mechanical system, steady state motion stabilization**

I. INTRODUCTION

The widespread use of relay controllers for technical devices and processes has led to the need to develop the qualitative theory of differential equations with discontinuous right-hand side [1]–[5]. Numerous studies have been devoted to the stability problem of the solutions for such equations. Let us highlight the main publications in our opinion which deal with a solution to this problem in the study direction of the presented paper. Various aspects on the generalization of classical Lyapunov theorems for equations with a discontinuous right-hand side and differential inclusions are considered in [3], [6]–[9]. The asymptotic stability theorem for such equations in an autonomous case is proved in [10], when Lyapunov function with a semi-definite time derivative exists.

In the papers [11]–[14], theorems on the application of Lyapunov function with a semi-definite time derivative to the asymptotic stability problem of non-autonomous differential equations with a discontinuous right-hand side are proved.

The development of the direct Lyapunov method in terms of the use of semi-definite Lyapunov functions in the stability study for continuous non-autonomous differential equations [15]–[17] made it possible to obtain new methods for solution to the control problems of mechanical systems [18]–[20].

The aim of this paper is to obtain new results in the direction presented in [11], [12], [21] and to solve on their basis the problems of applying the relay controllers to stabilize the steady state motions of controlled mechanical systems.

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The remainder of the paper is organized as follows. Section II presents the necessary for the following study qualitative properties of an equation with a discontinuous right-hand side. In Section III, theorem on the quasi-invariance of the positive limit set of a bounded solution of such an equation has been solved under the assumption of the Lyapunov function with a semi-definite time derivative existence. On the basis of this theorem, in Section IV, new theorems on sufficient conditions for the asymptotic stability of the zero solution have been proved. Section V presents the solution to the steady state motion stabilization problem of a holonomic mechanical system with cyclic coordinates. In Section VI, the steady state motion stabilization problem for a five-link robot manipulator has been solved.

II. PRELIMINARY NOTIONS

Let R^n be an n-dimensional real linear space. Let $(\cdot)'$ be a transpose operation. Let $x = (x_1, x_2, \dots, x_n)'$ be a vector of R^n. Denote by $|x|$ the vector norm in R^n, $R = (-\infty, \infty)$, $R^+ = [0, \infty)$.

Consider the differential equation

$$\dot{x} = g(t, x), \tag{1}$$

where the right-hand side is the function g defined in some domain $R \times D$, the set $D \subset R^n$ can be represented as $D = D_0 \cup M$, $D_0 = D_1 \cup D_2 \cup \ldots \cup D_l$. The sets D_i $(i = 1, 2, \dots, l)$ are disjoint subdomains of D. The set M of zero measure consists of the boundaries of D_i $(i = 1, 2, \dots, l)$. In each subdomain $R \times D_j$ $(j = 1, 2, \dots, l)$, the function g is continuous.

Assume that for each fixed point $t \in R$, the function g has a finite limit, i.e. $g(t, x_k) \to g_0 = constant$ as $x_k \to x_0 \in M$, where the value of the vector g_0 depends on the choice of the sequence $x_k \to x_0$. Thus, at each time point $t \in R$, M is the set of discontinuity of the function g.

Let for each point $(t, x) \in R \times D$, $G(t, x)$ be the smallest convex closed set containing all limit values of the function $g(t, x_k)$, where $x_k \in D_0$, $x_k \to x$ as $k \to \infty$.

Definition 1. [3] *Solution of (1) is a solution to the differential inclusion*

$$\dot{x} \in G(t, x) \tag{2}$$

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which is an absolutely continuous function $x(t)$ defined on some interval (α, β) such that the inclusion

$$\dot{x}(t) \in G(t, x(t))$$

holds almost everywhere for $t \in [a, b] \subset (\alpha, \beta)$.

An important method of modelling the systems (1) is the extension of the function g such that the inclusion (2) is suitable for an approximate description of the processes in the real world systems.

It is convenient to determine when the right-hand side of (1) has the form [1], [3]

$$g = g(t, x, u_1(t, x), u_2(t, x), \dots, u_r(t, x)), \tag{3}$$

where the function $g = g(t, x, u_1, u_2, \dots u_r)$ is continuous in all its variables, and scalar functions $u_j(t, x)$ $(j = 1, 2, \dots, r)$ are piecewise continuous in the domain $R \times D_0$, i.e. every function $u_j(t, x)$ is continuous on the set $R \times D$ except for the set M of the zero measure for fixed $t \in R$, while the function $u_j(t, x)$ has finite limits as $(t_k, x_k) \to (t, x) \in R \times M_j$, and the set of all limiting values of $u_j(t, x)$ is convex.

A multi-valued function

$$g(t, x) = g(t, x, U_1(t, x), \dots, U_r(t, x))$$

can be introduced such that for every point (t, x), the function $g(t, x)$ has the values defined by independent changes of $U_1(t, x), \dots, U_r(t, x)$.

Assume that a scalar function $u_j(t, x)$ is discontinuous on the smooth surface

$$S_j = \{x \in D : \ \psi_j(x) = 0, \ \psi_j \in C^1(D)\}.$$

Denote by M the following union

$$M = \bigcup_{j=1}^{m} S_j \quad (m \le r).$$

At the points belonging to one or several surfaces at the same time, for example the surfaces $S_1, S_2, \dots, S_m$, one can assume the following (if a solution of (1) cannot immediately leave such a surface or the intersection of these surfaces)

$$\dot{x} = g(t, x, u_1^{eq}(t, x), \dots, u_m^{eq}(t, x), u_{m+1}(t, x), \dots, u_r(t, x)), \tag{4}$$

where the equivalent controllers $u_1^{eq}, \dots, u_m^{eq}$ are defined so that the vector g in (3) touches the surfaces $S_1, S_2, \dots, S_m$ and so that the value $u_i^{eq}(t, x)$ is contained in the segment with the endpoints $u_i^-(t, x)$ and $u_i^+(t, x)$, where u_i^- and u_i^+ are the limit values of the function u_i on the both sides of the surface S_i, $i = 1, \dots, m$.

So, the functions $u_i^{eq}(t, x)$ $(i = 1, \dots, m)$ are defined from the following system of equations

$$\begin{gathered}\nabla\psi_i(x) \cdot g(t, x, u_1^{eq}, u_2^{eq}, \\ \dots, u_m^{eq}, u_{m+1}(t, x), \dots, u_r(t, x)) = 0, \ i = 1, 2, \dots, m.\end{gathered} \tag{5}$$

Definition 2. [3] *Solution of (4) is an absolutely continuous function $x = x(t)$ which outside the surfaces S_j $(j = 1, \dots, m)$ satisfies (1), (3), and on these surfaces and their intersections satisfies (4).*

Assume that the function g in (1), (3) is linear on $u_1, u_2, \dots, u_r$; all the surfaces S_j $(j = 1, 2, \dots, m)$ are distinct, and at the points of their intersection, the normal vectors are linearly independent. In this case, Definitions 1 and 2 are the same ones [3].

Further, when studying the control problems, assume that the following holds.

For two non-empty closed sets A and B from R^n, define the value $\beta(A, B) = \sup(\rho(a, B), a \in A)$, where $\rho(a, B) = \inf(|b - a|, b \in B)$. Following [3], assume that the multivalued function $G(t, x)$ in the domain $R \times D$ satisfies the following conditions: for all $(t, x) \in R \times D$, the set $G(t, x)$ is non-empty, bounded, closed, convex, and the function $G(t, x)$ is $\beta-$continuous in (t, x) which means that $\beta(G(t, x), G(t_0, x_0)) \to 0$ as $(t, x) \to (t_0, x_0)$. Note that this condition is sufficient for the existence of a solution of (1) for each initial point $(t_0, x_0) \in R \times D$.

Assume that the right-hand side of (1) satisfies the conditions: for each compact sets $K \subset D$ and $K_j \subset D_j$ $(j = 1, 2, \dots, l)$ there exist the constants $m = m(K)$ and $L_j = L_j(K_j)$ $(j = 1, 2, \dots, l)$ such that

$$|g(t, x)| \le m, \ |g(t, x_2) - g(t, x_1)| \le L_j|x_2 - x_1| \tag{6}$$

for all $(t, x) \in R \times K$ and all $(t, x_1), (t, x_2) \in R \times K_j$.

According to [22], [23], one can construct a space of functions $F : R \times D_0 \to R^n$ wherein the family of translates $\{g_\tau(t, x) = g(\tau + t, x), \ \tau \in R, \ (t, x) \in R \times D_0\}$ is precompact. Accordingly, for any sequence $t_l \to \infty$, there exist both a subsequence $\{t_k\} \subset \{t_l\}$ and a function $g^* \in F$ such that

$$g^*(t, x) = \frac{d}{dt} \lim_{k\to\infty} \int_0^t g(t_k + \tau, x) d\tau.$$

Then, in each domain $R \times D_j$ one can construct a family of limiting equations [23]

$$\dot{x} = g^*(t, x), \ g^*(t, x) = \frac{d}{dt} \lim_{t_k\to\infty} \int_0^t g(t_k + \tau, x) d\tau. \tag{7}$$

Let construct a set of limiting equations (7) for each subdomain $D_j \subset D$ $(j = 1, 2, \dots, l)$ and define the general set of limiting equations for the domain $D_0 = D_1 \cup D_2 \cup \dots \cup D_l$ according to the following definition.

Definition 3. *Equation (7) defined in the domain $D_0 = D_1 \cup D_2 \cup \dots \cup D_l$ is called limiting one to (1) in relation to the sequence $t_k \to +\infty$, if it is defined as the limit to (1) for this sequence in each domain D_j $(j = 1, 2, \dots, l)$.*

Let some equation (7) be a limiting one for (1) with respect to the sequence $t_k \to \infty$. Similarly to the previous one, for each point $(t, x) \in R \times D$ define by $G^*(t, x)$ the smallest convex set containing the sets $\{g(t_k + t, x_k) : \ t_k \to \infty, \ x_k \to x \in D\}$ and $\{g^*(t, x_k) : \ x_k \to x \in D\}$.

Define the limiting inclusion as follows

$$\dot{x} \in G^*(t,x). \tag{8}$$

Definition 4. *Similarly to Definitions 1 and 2, we call $x = x^*(t)$ a solution to the limiting inclusion (8), if this function is an absolutely continuous one on the interval (α, β) and is such that*

$$\dot{x}(t) \in G^*(t, x(t))$$

for almost all $t \in [a,b] \subset (\alpha, \beta)$.

Definition 5. *Let $x = x(t)$ be a solution of (1) defined for all $t \geq t_0$. A point $p \in D$ is called a limiting point for $x = x(t)$, if there exists a sequence $t_k \to \infty$ such that $x(t_k) \to p$ as $k \to \infty$. The set $\omega^+(x(t))$ of all such points is a positive limit set.*

Theorem 1. *Let $x = x(t)$ be some solution of (1) according to Definition 1, bounded for all $t \geq t_0$ by some compact set $K \subset D$, i.e. $\{x(t),\ t \geq t_0\} \subset K$. Then, the positive limit set $\omega^+(x(t))$ of this solution is weakly quasi-invariant, namely, for each point $p \in \omega^+(x(t))$, there exist both a limiting inclusion (8) and some of its solution $x = x^*(t)$ such that $x^*(0) = p$, $\{x^*(t),\ t \in R\} \subset \omega^+(x(t))$.*

Proof. Let the limit point $p \in \omega^+(x(t))$ be defined by the sequence $t_j \to \infty$. From the condition (6), one can see that the solution $x = x(t)$ is a uniformly continuous function in $t \in [t_0, \infty)$. Hence, there exist both the subsequence $\{t_k \to \infty\ k \to \infty\} \subset \{t_j \to \infty\ j \to \infty\}$ and the function $x = x^*(t)$, $t \in R$ such that $x_k(t) = x(t_k + t) \to x^*(t)$ as $k \to \infty$ uniformly in $[-T, T]$ for every $T > 0$. Obviously, the function $x^*(t)$ is absolutely continuous in $t \in [a,b] \subset R$, $x^* \in \omega^+(x(t))$ for all $t \in R$.

Without loss of generality, one can assume that the sequence $\{g_k(t,x) = g(t_k+t, x)\}$ converges to $g^* \in F$ as $t_k \to \infty$. Let us define the corresponding multivalued function $G^*(t,x)$.

The following cases are possible.

Case 1. For the function $x = x^*(t)$, the following holds $x^*(t) \in D_j$ for $t \in (\alpha, \beta) \subset R$.

Then, for sufficiently large k one can obtain that the solution $x = x(t)$ of (1) satisfies

$$x_k(t) = x_k(\gamma) + \int_\gamma^t g_k(\tau, x_k(\tau))d\tau,\ \alpha_1 < \gamma < \beta_1$$
$$g_k(t,x) = g(t_k + t, x)$$

for all $t \in (\alpha_1, \beta_1) \subseteq (\alpha, \beta)$.

From this, passing to the limit as $k \to \infty$, one can obtain that $x = x^*(t)$ is a solution of (7) for $t \in (\alpha, \beta)$.

Case 2. The function $x = x^*(t)$ is such that $x^*(t) \in M$ for $t \in [\alpha, \beta]$, ($\alpha \leq \beta$).

For the sequence $x_k(t)$, the following equality holds

$$x_k(t) = x_k(\gamma) + \int_\gamma^t G_k(\tau, x_k(\tau))d\tau, \quad \alpha < \gamma < \beta_1,$$
$$G_k(t,x) = G(t_k + t, x)$$

for all $t \in (\alpha_1, \beta_1) \supset [\alpha, \beta]$.

On the base from [2], [13], one can find that the convergence of $G_k(t,x) \to G^*(t,x)$ implies that $\dot{x}^*(t) \subset G(t, x^*(t))$, $t \in (\alpha_1, \beta_1)$.

Thus, $x = x^*(t)$ is a solution of (8) defined for all $t \in R$. Theorem 1 is proved.

III. Quasi-Invariance Principle

For the function $V(t,x) \in C^1(R^+ \times D)$ define the upper derivative by virtue of (1) using the equality [3]

$$\dot{V}^* = \left(\frac{dV}{dt}\right)^* = \sup_{y \in G(t,x)} \left(V_t + \sum_{i=1}^n \frac{\partial V}{\partial x_i} y_i\right), \tag{9}$$

where $V_t = \partial V(t,x)/\partial t$.

Accordingly, for each solution $x = x(t)$ of (1) there exists a time derivative of $V(t,x)$ such as

$$\dot{V} = \frac{d}{dt} V(t, x(t)) = V_t + \sum_{i=1}^n \frac{\partial V}{\partial x_i} \dot{x}_i(t),$$

and this derivative satisfies the inequality $\dot{V} \leq \dot{V}^*$.

Assumption 1. *Assume that the derivative $\dot{V}^*(t,x)$ defined by (9) for all $(t,x) \in R^+ \times D_H$ is estimated by the inequality*

$$\dot{V}^*(t,x) \leq -W(t,x) \leq 0, \tag{10}$$

where the function $W(t,x)$ is bounded and uniformly continuous on the set $R^+ \times K$, namely, for each compact set $K \subset D$ there exists $m = m(K) > 0$ and for each small $\varepsilon > 0$ one can find $\delta = \delta(\varepsilon, K) > 0$ such that for all $(t,x), (t_1, x_1), (t_2, x_2) \in R^+ \times K$ satisfying the relationships

$$|t_1 - t_2| < \delta,\ |x_1 - x_2| < \delta$$

the following inequalities hold

$$W(t,x) \leq m(K),\ |W(t_2, x_2) - W(t_1, x_1)| < \varepsilon.$$

Introduce the family of limiting functions by means of the relation [24]

$$\lim_{t_{kl} \to \infty} W(t_{kl} + t, x) = W^*(t,x).$$

Definition 6. *Let $t_k \to \infty$ be some sequence, and let $t \in R$, $c \in R$ be some constants. Define the set $V^{-1}(t,c)$ as follows*

$$V_\infty^{-1}(t,c) = \{x \in D:\ \exists x_k \to x,\ V(t_k + t, x_k) \to c\}.$$

Definition 7. [15] *The limiting to (1) equation (8) and the limiting to $W = W(t,x)$ function $W^* = W^*(t,x)$ combine the limiting pair (G^*, W^*), if they are the limiting ones for the same sequence $t_k \to +\infty$. Functions G^* and W^* form a limiting pair (G^*, W^*), if they are limiting for the same sequence $t_k \to \infty$. The set $V_\infty^{-1}(t,c)$ defined by the same sequence $t_k \to \infty$ is said to be corresponding to the pair (G^*, W^*).*

Theorem 2 (Quasi-invariance principle). *Assume that:*

1) for all $t \geq t_0$, the solution $x = x(t)$ of (1) is bounded by some compact $K \subset D$;

2) Lyapunov function exists $V \in C^1(D)$, $V(t,x) \geq m(K)$ for all $(t,x) \in R \times K$, whose time derivative satisfies the inequality (10).

Then, for each limit point $p \in \omega^+(x(t))$, there exist both a limiting pair (G^*, W^*) and some solution $x = x^*(t)$, $x^*(0) = p$ of inclusion (8) such that $\{x^*(t),\ t \in R\} \subset \omega^+(x(t))$, $W^*(t, x^*(t)) \equiv 0$, $x^*(t) \in V_\infty^{-1}(t, c_0)$ for some constant c_0.

Proof. It follows from the conditions of the theorem that for the solution $x = x(t)$ of (1) there exists a real $c = c_0 \geq m(K)$ such that

$$V(t, x(t)) \to c_0 \text{ as } t \to \infty. \tag{11}$$

Let the limit point $p \in \omega^+(x(t))$ be defined by the sequence $t_k \to \infty$. Let (G^*, W^*) be the corresponding limiting pair. According to Theorem 1, one can assume that the following holds $x_k(t) = x(t_k + t) \to x^*(t)$, $x^*(0) = p$ as $t_k \to +\infty$. Moreover, $x = x^*(t)$ is some solution of the corresponding inclusion (8) such that

$$\{x^*(t),\ t \in R\} \subset \omega^+(x(t)).$$

From (11), it follows that for each $t \in R$

$$V(t_k + t, x_k(t)) = V(t_k + t, x(t_k + t)) \to c_0.$$

Therefore, $x^*(t) \in V_\infty^{-1}(t, c_0)$ for all $t \in R$.

From condition 2 of Theorem 2 one can also find the inequality

$$V(t_k + t, x_k(t)) - V(t_k, x_k(0)) \leq -\int_0^t W_k(\tau, x_k(\tau))d\tau \leq 0, \tag{12}$$

where $W_k(\tau, x) = W(t_k + \tau, x)$.

Passing to the limit in (12) as $t_k \to +\infty$, one can obtain that

$$W^*(t, x^*(t)) \equiv 0.$$

Thus, Theorem 2 is proved.

IV. Stability Theorems

Let $D = \{x \in R^n : |x| < d,\ 0 < d \leq +\infty\}$. Assume that the function $g(t, x)$ satisfies the condition $g(t, 0) \equiv 0$. Thus, the equation (1) and inclusion (8) have the zero solution $x = 0$.

The following result holds, where $a_1, a_2 \in \mathcal{K}$ ($\mathcal{K}$ is a class of continuous functions $a_i :\ R^+ \to R^+$ such that $a_i(0) = 0$ and $a_i(y)$ is strictly increasing for all $y \in R^+$ [25]).

Theorem 3. *Assume that: 1) there exists a positive definite function $V = V(t, x)$, i.e.*

$$V(t, x) \geq a_1(|x|);$$

2) the time derivative of the function V by virtue of (1) satisfies

$$\dot{V} \leq -W(t, x) \leq 0;$$

3) for each limiting pair (G^*, W^*), the set $\{V_\infty^{-1}(t, c) : c = c_0\} \cap \{W^*(t, x) = 0\}$ does not contain any solutions of $\dot{x} \in G^*(t, x)$ except for $x = 0$.

Then, the zero solution $x = 0$ of (1) is weakly asymptotically stable.

Theorem 4. *Assume that: 1) there exists a positive definite function $V = V(t, x)$ admitting an infinitely small upper limit, i.e.*

$$a_1(|x|) \leq V(t, x) \leq a_2(|x|);$$

2) the time derivative of the function V by virtue of (1) satisfies

$$\dot{V} \leq -W(t, x) \leq 0;$$

3) for each limiting pair (G^*, W^*) the set $\{W^*(t, x) = 0\}$ does not contain any solutions of $\dot{x} \in G^*(t, x)$ except for $x = 0$.

Then, the zero solution $x = 0$ of (1) is uniformly asymptotically stable.

Choose some compact set $K_0 = \{x \in R^n :\ |x| \leq d_0 < d\}$ and introduce the set Φ of the functions $\varphi : R^+ \to K_0$ such that

$$|\varphi(t_2) - \varphi(t_1)| \leq L(K_0)|t_2 - t_1|\ \ \forall t_1, t_2 \in R^+.$$

Obviously, the function φ is absolutely continuous, and if the function $\varphi^* :\ R^+ \to K_0$ is limiting for $\{\varphi_k(t) = \varphi(t_k + t),\ t_k \to +\infty,\ t \geq 0\}$, then the function $\varphi^*(t) \in \Phi$ is also absolutely continuous.

For the vector $\psi\ :\ R^n \to R^m$ defining the surface S_j ($j = 1, 2, \ldots, m$), introduce the norm of ψ as follows

$$\|\psi\|^2 = \sum_{j=1}^m \psi_j^2.$$

Definition 8. *The zero state $x = 0$ is uniformly asymptotically stable with respect to both the set of functions Φ and the set $\{x \in K_0 :\ \psi(x) = 0\}$, if :*

1) $(\forall \varepsilon > 0)$ $(\exists \delta = \delta(\varepsilon) > 0)$ $(\forall x_0 : |x_0| < \delta)$ $(\forall \varphi \in \Phi : \varphi(0) = x_0)$ $(\forall t \geq 0)$ $|\varphi(t)| < \varepsilon$;

2) $(\exists \Delta,\ 0 < \Delta < d_0)$ $(\forall \varepsilon > 0)$ $(\exists T = T(\varepsilon) > 0)$ $(\forall x_0 : |x_0| \leq \Delta)$ $(\forall \varphi \in \Phi, \varphi(0) = x_0)$ $(\forall t \geq T)$ $|\varphi(t + T)| < \varepsilon$.

Theorem 5. *Under conditions 1 and 2 of Theorem 1, we also assume that:*

3) for each limiting pair (g^*, W^*), the set

$$\{W^*(t, x) = 0\} \cap \{x \in R^n :\ |x| \geq \mu > 0\} \cap D_0$$

does not contain any solutions of the equation $\dot{x} = g^*(t, x)$;

4) the zero state $x = 0$ is uniformly asymptotically stable with respect to both the set of functions Φ and the set $\{x \in K_0 :\ \psi(x) = 0\}$.

Then, the zero solution $x = 0$ of (1) is asymptotically stable.

Proof. *From conditions 1 and 2 of Theorem 1 it follows that the solution $x = 0$ of (2) is uniformly stable [7].*

Let $x = x(t)$, $x(t_0) = x_0$ be some solution of (2) bounded by the compact K_0 for all $t \geq t_0$. Let $V(t) = V(t, x(t))$ be Lyapunov function candidate along the solution $x(t)$. Define the set $\omega^+(x(t))$ of all limit points of the solution $x(t)$ as follows

$$\omega^+(x(t)) = \{p \in K_0 :\ \exists t_k \to \infty,\ x(t_k) \to p\}.$$

Let $p \in \omega^+(x(t)) \cap D_0$. Then, there exist both a limiting equation (7) and its solution $x = \varphi(t)$, $\varphi(0) = p$ such as

$$\{\varphi(t) : \ t \in [0, \gamma), \ \gamma > 0\} \subset D_0.$$

From the results in [15], [16], it can be shown that by virtue of the conditions 1, 2, and 3 of Theorem 2, the following holds $V(t) \to 0$ as $t \to \infty$.

Assume otherwise $\omega^+(x(t)) \subset \{\psi = 0\}$. Then, for each point $p \in \omega^+(x(t))$, there exists a function $\varphi^* \in \Phi$ such as $x(t_k + t) \to \varphi^*(t)$ uniformly on $t \in [0, T]$ for each $T > 0$ as $t_k \to \infty$.

But the condition 4 of the theorem implies that $a_2(|\varphi^*(t)|) \to 0$ as $t \to \infty$. By the condition 1 of Theorem 1 one can also obtain that $a_2(|x(t)|) \to 0$ as $t \to +\infty$.

Thus, along each solution $x(t)$ of (2) such that $x(t) \in K_0$ one can obtain that $|x(t)| \to 0$ as $t \to \infty$. Accordingly, the solution $x = 0$ of (2) is asymptotically stable. Thus, one can get the end of the proof.

Theorem 6. *Assume that one can find the Lyapunov function candidate $V = V(t, x)$ satisfying the conditions:*

1) $0 \le V(t, x) \le \|\psi(x)\|^2$;

2) $\dot{V}^*(t, x) \le -\|\psi(x)\|$;

3) the zero state $x = 0$ of (1) is uniformly asymptotically stable with respect to the motions of (1) or it is lying on the set $\{\psi(x) = 0\}$.

Then, the zero solution $x = 0$ of (1) is uniformly asymptotically stable.

Note that the proof of Theorem 6 is similar to one of Theorem 5.

V. On the Control Problem of Mechanical Systems with Cyclic Coordinates

Let us consider a mechanical system with holonomic stationary constraints described by n generalized coordinates wherein m coordinates $r = (r_1, r_2, \ldots, r_m)'$ $(1 \le m < n)$ are positional, the rest $(n - m)$ coordinates $s = (s_1, s_2, \ldots, s_{n-m})'$ are cyclic.

The kinetic energy T of the mechanical system can be written as follows

$$2T = \dot{r}' A_{rr}(r)\dot{r} + \dot{r}' A_{rs}(r)\dot{s} + \dot{s}' A_{sr}(r)\dot{r} + \dot{s}' A_{ss}(r)\dot{s},$$

where $A_{rr}(r) \in R^{m\times m}$, $A'_{sr}(r) = A_{rs}(r) \in R^{m\times(n-m)}$, and $A_{ss}(r) \in R^{(n-m)\times(n-m)}$ are the blocks of the mechanical system's inertia matrix.

By using the so-called Routhian which is the vector of conjugate momenta $p = (p_1, p_2, \ldots, p_{n-m})'$ corresponding to the cyclic coordinates s, one can obtain the motion equations of the mechanical system as Routh equations [26], [27]

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = -\frac{\partial \Pi}{\partial r} + Q_r + U_r, \\ \frac{dp}{dt} = Q_s + U_s, \ \dot{s} = -\frac{\partial R}{\partial p}, \end{gathered} \tag{13}$$

where R is the Routh function, $R = R_2 + R_1 + R_0$, $2R_2 = \dot{r}' B_2(r)\dot{r}$, $B_2(r) = A_{rr}(r) - A_{rs}(r)A_{ss}^{-1}(r)A_{sr}(r)$, $R_1 = p' B_1(r)\dot{r}$, $B_1(r) = A_{ss}^{-1}(r)A_{sr}(r)$, $2R_0 = -p' B_0(r)p$, $B_0(r) = A_{ss}^{-1}(r)$, $\Pi = \Pi(r)$ is the potential energy, $Q = (Q_r, Q_s)'$ is the vector of generalized forces, $U = (U_r, U_s)'$ is the vector of control inputs.

Consider the case when the forces Q_r are a combination of gyroscopic and dissipative forces, $Q_r = Q_r(t, r, \dot{r})$, $Q_r(t, r, 0) = 0$, $Q'_r\dot{r} \le 0$, and the forces Q_s are perturbing ones which are bounded, i.e. $Q_s = Q_s(t, q, \dot{q})$, $|Q_s(t, q, \dot{q})| \le h(t) \le h_0$.

If $Q_s \equiv 0$, $U_r = 0$, $U_s = 0$, then the equalities

$$\frac{\partial}{\partial r}(R_0 - \Pi) \equiv 0$$

define the perturbed solutions of the system (13) as follows

$$\begin{gathered} \dot{r} = 0, \ r = r^{(0)} = constant, \ p = p^{(0)} = constant, \\ \dot{s} = \dot{s}^{(0)} = B_0(r^{(0)})p^{(0)}, \end{gathered} \tag{14}$$

which are called the stationary motions [26], [27].

Consider the solution to the stationary motion stabilization problem.

Solution 1. Construct the controller

$$U_r = -F(t, r)\dot{r} - \frac{\partial \Pi_U(r)}{\partial r}, \tag{15}$$

$$\begin{gathered} U_s = (-\mu_1 \operatorname{sign}(\dot{s}_1 - \dot{s}_1^{(0)}), -\mu_2 \operatorname{sign}(\dot{s}_2 - \dot{s}_2^{(0)}), \\ \ldots, -\mu_{n-m} \operatorname{sign}(\dot{s}_{n-m} - \dot{s}_{n-m}^{(0)}))', \end{gathered} \tag{16}$$

where the functions $F(t, r)$ and $\Pi_U(r)$ satisfy the following relationships

$$\begin{gathered} \dot{r}' F(t, r)\dot{r} \ge f_0\|\dot{r}\|^2, \ f_0 > 0, \ \mu_j > h_0, \\ j = 1, 2, \ldots, n - m, \\ \left.\frac{\partial \Pi_U(r)}{\partial r}\right|_{r=r^{(0)}} = 0, \\ \left\|\frac{\partial}{\partial r}(R_0 - \Pi - \Pi_U)\right\| \ge \delta(\varepsilon) > 0 \\ \text{as } p = p^{(0)}, \ \|r - r^{(0)}\| = \varepsilon > 0. \end{gathered} \tag{17}$$

Choose the Lyapunov function candidate such as

$$V(\dot{r}, r, p) = R_2 + \Pi + \Pi_U - R_0 + \left.\left(\frac{\partial R_0}{\partial p}\right)'\right|_{p=p^{(0)}} (p - p^{(0)}). \tag{18}$$

For the time derivative of (18) by virtue of (13) one can find the estimation

$$\dot{V} \le -f_0\|\dot{r}\|^2 - \sum_{j=1}^{n-m} (\mu_j - h_0)|\dot{s}_j - \dot{s}_j^{(0)}|.$$

In accordance with Theorem 4, one can obtain the following solution to the problem.

Statement 1. *Let the function $V(\dot{r}, r, p) - V(0, r^{(0)}, p^{(0)})$ is positive definite with respect to $(\dot{r}, r - r^{(0)}, p - p^{(0)})$. Then, the controller (15), (16) provides the uniform asymptotic stability of the motion (14).*

Solution 2. *Assume that $U = (U_r', U_s')'$ is such that the controller U_r is defined by (15) and*

$$
\begin{aligned}
U_s = (&-\mu_1 \operatorname{sign}(p_1 - p_1^{(0)}), -\mu_2 \operatorname{sign}(p_2 - p_2^{(0)}), \\
&\ldots, -\mu_{n-m} \operatorname{sign}(p_{n-m} - p_{n-m}^{(0)}))'
\end{aligned}
\tag{19}
$$

and the conditions (17) hold.

To derive this solution, we define two Lyapunov functions candidates

$$
V_1 = \|p - p^{(0)}\|^2, \quad V_2 = R_2 + V_3,
$$

where $V_3(r, p^{(0)}) = \Pi(r) - \Pi_U(r) - R_0(r, p^{(0)})$.

For the time derivative of the first function V_1, one can find

$$
\dot V_1 \le -2 \sum_{j=1}^{n-m} (\mu_j - h_0)|p_j - p_j^{(0)}| \le 0.
$$

For the time derivative of the second function V_2, one can obtain

$$
\dot V_2 \le -f_0 \|\dot r\|^2 \le 0.
$$

Using Theorem 6, one can obtain the following statement.

Statement 2. *Let the function $V_3(r, p^{(0)}) - V_3(r^{(0)}, p^{(0)})$ be positive definite with respect to $p - p^{(0)}$. Then, the controller (15), (19) provides the uniform asymptotic stability property for the motion (14).*

VI. Solution to the Stationary Motion Stabilization Problem for a Five-Link Robotic Manipulator

The model of a five-link robotic manipulator (see Fig.1) can described as follows. Each link of the manipulator is represented as a solid body. The kinematic pairs of the manipulator are assumed to be single-link, their geometric centers are denoted by O_k $(k = 1, 2, \ldots, 5)$. The centers of mass C_k $(k = 1, 2, \ldots, 5)$ of the links lie on the axes $O_k O_{k+1}$, the axes $O_k O_{k+1}$ $(k = 1, 2, \ldots, 5)$ are their axes of symmetry. The first base link is vertical, it rotates around $O_1 O_2$, its rotation angle is θ_1. The second kinematic pair allows rotation of the second link around the horizontal axis passing through O_2. The third kinematic pair allows rectilinear movement of the third link along $O_2 O_4$ $(O_3 \in O_1 O_4)$, let us introduce the notation for the displacement of the third link $x = d_3 = O_1 O_4$. The fourth link can rotate around a horizontal axis passing through O_4 with the angle of rotation θ_4. The fifth link simulating the gripper allows the rotation around $O_4 O_5$, its rotation angle is denoted by θ_5.

We will assume that the centers of mass C_k of the links lie on the axes $O_k O_{k+1}$ and these axes are the axes of symmetry of the corresponding links $(k = 1, 2, \ldots, 5)$. Let's introduce the main central axes $C_k x_k, C_k y_k, C_k z_k$ of the links. We will assume that for links 1 and 5 the axes $C_1 z_1$ and $C_5 z_5$ are the axes of symmetry. For links 2, 3 and 4 such axes are $C_2 x_2$, $C_3 x_3$ and $C_4 x_4$ respectively. Let us assume that the axes $C_2 z_2$, $C_3 z_3$ and $C_4 z_4$ are horizontal. The masses of the links are denoted by m_k $(k = 1, 2, \ldots, 5)$, and their main central moments of inertia are denoted by $I_x^{(k)}$, $I_y^{(k)}$ and $I_z^{(k)}$.

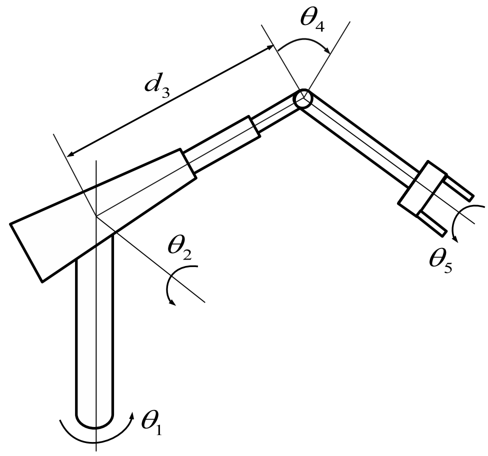

Fig. 1. Model of a five-link robotic manipulator.

Accordingly, we have $I_x^{(1)} = I_y^{(1)}$, $I_x^{(5)} = I_y^{(5)}$, $I_y^{(2)} = i_z^{(2)}$, $I_y^{(3)} = I_z^{(3)}$, $I_y^{(4)} = I_z^{(4)}$. Let's introduce the lengths $O_2 C_2 = l_2$, $C_3 O_4 = l_3$, $O_4 O_5 = 2 O_4 C_4 = 2 l_4$, and $O_5 C_5 = l_5$.

By using Koenig theorem we can find the kinetic energy T_i of each link $i = 1, 2, \ldots, 5$ as the kinetic energy of an absolutely rigid body.

$$
T_1 = \frac{1}{2} I_z^{(1)} \dot\theta_1^2,
$$

$$
\begin{gathered}
T_2 = \tfrac{1}{2} m_2 l_2^2 (\sin^2\theta_2 \dot\theta_1^2 + \dot\theta_2^2) \\
+ \tfrac{1}{2} (I_x^{(2)} \dot\theta_1^2 \cos^2\theta_2 + I_z^{(2)} (\dot\theta_1^2 \sin^2\theta_2 + \dot\theta_2^2)),
\end{gathered}
$$

$$
\begin{gathered}
T_3 = \tfrac{1}{2} m_3 (\dot x^2 + (x - l_3)^2 \dot\theta_z^2 + (x - l_3) \sin^2\theta_2 \cdot \dot\theta_1^2) \\
+ \tfrac{1}{2} (I_x^{(3)} \dot\theta_1^2 \cos^2\theta_2 + I_z^{(3)} (\dot\theta_1^2 \sin^2\theta_2 + \dot\theta_2^2)),
\end{gathered}
$$

$$
\begin{gathered}
T_4 = \tfrac{1}{2} m_4 ((\dot x + l_4 \dot\theta_4 \sin\theta_4)^2 + (x \dot\theta_2 - l_4 \dot\theta_4 \cos\theta_4)^2 \\
+ (x \cos\theta_4 - l_4 \sin(\theta_2 + \theta_4))^2 \dot\theta_1^2 \\
+ \tfrac{1}{2} (I_x^{(4)} \dot\theta_1^2 \cos^2(\theta_2 + \theta_4) \\
+ I_z^{(4)} (\dot\theta_1^2 \sin^2(\theta_2 + \theta_4) + (\dot\theta_2 + \dot\theta_4)^2)),
\end{gathered}
$$

$$
\begin{gathered}
T_5 = \tfrac{1}{2} m_5 ((\dot x + (2l_4 + l_5) \dot\theta_4 \sin\theta_4)^2 \\
+ (x \dot\theta_2 - (2l_4 + l_5) \dot\theta_4 \cos\theta_4)^2 \\
+ (x \cos\theta - (2l_4 + l_5) \sin(\theta_2 + \theta_4))^2 \dot\theta_1^2 \\
+ \tfrac{1}{2} I_x^{(5)} (\dot\theta_1 \cos(\theta_2 + \theta_4) + \dot\theta_5)^2 \\
+ \tfrac{1}{2} I_z^{(5)} (\dot\theta_1^2 \cos^2(\theta_2 + \theta_4) + (\dot\theta_2 + \dot\theta_4)^2).
\end{gathered}
$$

The kinetic energy of the manipulator is given by

$$
T = T_1 + T_2 + T_3 + T_4 + T_5.
$$

The potential energy of the manipulator has the form

$$
\begin{gathered}
\Pi = -m_2 g l_2 \cos\theta_2 - m_3 g (x - l_3) \cos\theta_2 \\
- m_4 g (x \cos\theta_2 + l_4 \cos(\theta_2 + \theta_4)) \\
- m_5 g (x \cos\theta_2 + (2l_4 + l_5) \cos(\theta_2 + \theta_4)).
\end{gathered}
$$

The rotation angle of the first link θ_1 is a cyclic coordinate, since

$$
\frac{\partial T}{\partial \theta_1} = \frac{\partial \Pi}{\partial \theta_1} = 0.
$$

The manipulator has a steady state motion of the form (14) such as

$$\begin{gathered}\dot{\theta}_1 = \dot{\theta}_1^{(0)},\ \dot{\theta}_2 = \dot{\theta}_4 = \dot{\theta}_5 = 0,\ \dot{x} = 0,\\ \theta_2 = \theta_2^{(0)},\ \theta_4 = \theta_4^{(0)},\ \theta_5 = \theta_5^{(0)},\ x = x^{(0)}.\end{gathered} \tag{20}$$

In the motion (20), the first link rotates at a constant angular velocity around the vertical, the remaining links are relatively motionless.

According to Solution 1, for any disturbing torque $|Q_1| \le \mu_0 < \mu$, the rotation (20) is stabilized by the control torque

$$\begin{gathered}U_s = -\mu\,\mathrm{sign}(\dot{\theta}_1 - \dot{\theta}_1^{(0)}),\\ U_{r_j} = -\mu\,\mathrm{sign}(\dot{\theta}_j + k(\sin(\theta_j - \theta_j^{(0)})),\ j = 2,4,5\\ U_{r_3} = -\mu\,\mathrm{sign}(\dot{x} + k(\sin(x - x^{(0)}))\end{gathered} \tag{21}$$

where $\mu > 0$ and $k > 0$.

Choose the control parameters as

$$k = 1.4,\ \ \mu = 100.$$

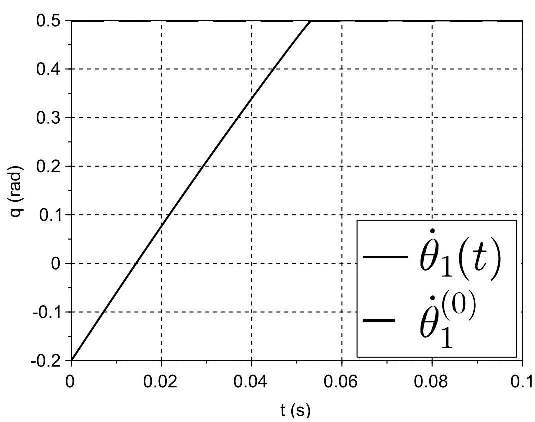

Fig. 2. Time evolution of the first link angular velocity.

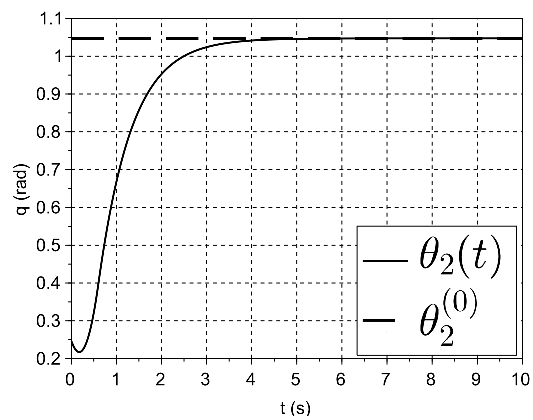

Fig. 3. Time evolution of the second link rotation angle.

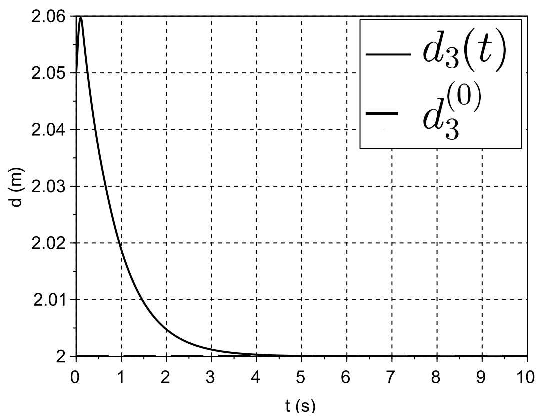

Fig. 4. Time evolution of the third link linear displacement

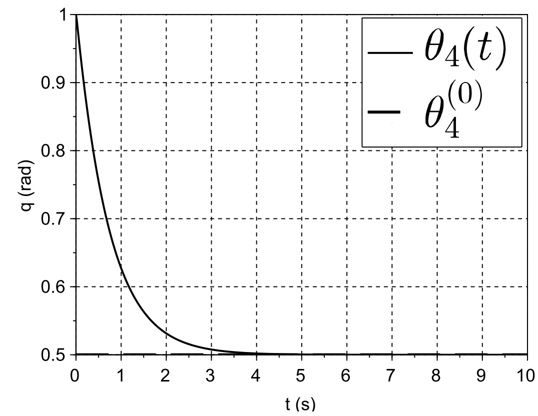

Fig. 5. Time evolution of the forth link rotation angle.

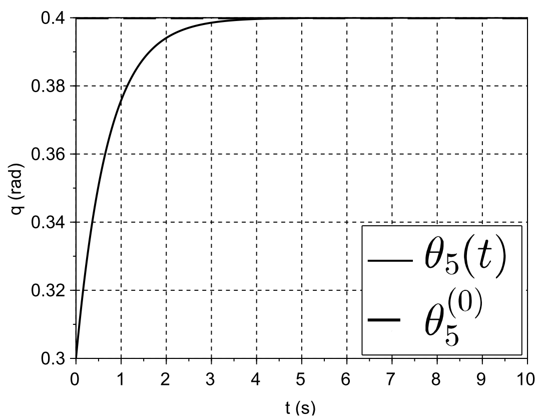

Fig. 6. Time evolution of the fifth link rotation angle.

The results of numerical simulation for the robot manipulator under the controller (21) show the steady state stabilization as can be seen in Fig. 2 – 6.

VII. Conclusion

The paper presents the development of the direct Lyapunov method in the stability study of the differential equations with discontinuous right-hand side on the basis of limiting equations and semi-definite Lyapunov functions. The development consists in the derivation of new theorems on the limiting behavior of solutions, the asymptotic stability of the zero solution. At the same time, the obtained structure of a limiting differential inclusion is convenient for modeling of controlled mechanical systems with relay controller. The effectiveness of this development is shown in solving the steady state motions stabilization problem of a holonomic mechanical system by relay controller. The problem of using relay controllers in the steady state motion stabilization of a five-link robot manipulator has been solved.

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