# Conservation Laws - a Source for Distortionless Propagation and Time Delays

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***Abstract*—Since the very first paper of J. Bernoulli in 1728, a connection exists between initial boundary value problems for hyperbolic Partial Differential Equations (PDE) in the plane (with a single space coordinate accounting for wave propagation) and some associated Functional Equations (FE). From the point of view of dynamics and control (to be specific, of dynamics *for* control) both type of equations generate dynamical and controlled dynamical systems. The functional equations may be difference equations (in continuous time), delay-differential (mostly of neutral type) or even integral/integro-differential. It is possible to discuss dynamics and control either for PDE or FE since both may be viewed as self contained mathematical objects.**

**A more recent topic is control of systems displaying conservation laws. Conservation laws are described by *nonlinear* hyperbolic PDE belonging to the class "lossless" (conservative); their dynamics and control theory is well served by the associated energy integral. It is however not without interest to discuss association of some FE. Lossless implies usually distortionless propagation hence one would expect here also lumped time delays.**

**The paper contains some illustrating applications from various fields: nuclear reactors with circulating fuel, canal flows control, overhead crane, drilling devices, without forgetting the standard classical example of the nonhomogeneous transmission lines for distortionless and lossless propagation. Specific features of the control models are discussed in connection with the control approach wherever it applies.**

***Index Terms*—conservation laws, distortionless propagation, time delays**

## I. Introduction and basics

We shall start from two elementary facts. First, any electrical or control engineer has dealt with mathematical models where either a complex domain term like $\mathrm{e}^{-\tau s}$ with $\tau > 0$, $s \in \mathbb{C}$, or a time domain term like $u(t-\tau)$, where $u$ was some signal, were present. Such models were called *time delay* or *time lag systems*. A more involved interest to such systems would inevitably have sent to some reference about the underlying equations of these models - the *equations with deviating argument*. A still more involved interest would have sent to the question concerning origins of these equations; interesting enough, the first differential equation with deviating argument, reported in [1], was published by Johann (Jean) Bernoulli in 1728 [2] and reads as

$$y'(t) = y(t-1) \tag{1}$$

As the title of this paper shows, this equation appears to be associated to a partial differential equation of hyperbolic type - the string equation. Even if the association seems to be mistaken in this paper, it sends nevertheless to the second elementary fact, less known, that propagation is associated to time delay. In order to explain this, we shall discuss a special case of propagation - the *lossless propagation*.By lossless propagation it is understood the phenomenon associated with long (in a definite sense) transmission lines for physical signals. In electrical and electronic engineering there are considered in various applications circuit structures consisting of multipoles connected through LC transmission lines (With respect to this a long list of references may be provided, starting with a pioneering paper of [3] and going up to a quite recent book of [4]). The lossless propagation occurs also for non-electric signals as water, steam or gas flows and pressures. With respect to this we may cite the pioneering (but almost forgotten) papers of [5], [6] on steam pipes for combined heat-electricity generation, the long list of papers dealing with waterhammer and many other. In order to illustrate these assertions, we shall consider one of the early benchmark problems, the nonlinear circuit containing a tunnel diode and a lossless transmission - the so called Nagumo-Shimura circuit (fig.1)

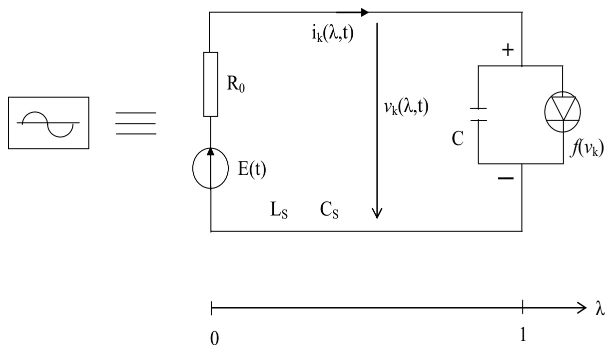


Fig. 1. Oscillator (Nagumo - Shimura circuit)

This circuit is described by the equations

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$$\begin{aligned}
&L_s\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial \lambda}\ ,\ C_s\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial \lambda},\ 0 \leq \lambda \leq 1 \\
&E = v(0,t) + R_0 i(0,t) \\
&-C\frac{\mathrm{d}}{\mathrm{d}t}\ v(1,t) = -i(1,t) + \psi(v(1,t))
\end{aligned} \tag{2}$$

Proceeding in “an engineering way” we may apply formally the Laplace transform to compute the solution of the boundary value problem viewed as independent of the differential equation but being nevertheless controlled by it. After some elementary but tedious formal manipulation we find the following time dependencies

$$\begin{aligned}
&v(t) + \sqrt{L/C}i(1,t) - \rho_0 v(t - 2\sqrt{LC}) + \\
&+\sqrt{L/C}i(1, t - 2\sqrt{LC})) = (1+\rho_0)E(t - \sqrt{LC}) \\
&C_0\frac{\mathrm{d}v}{\mathrm{d}t} + \psi(v) = i(1,t) \\
&\rho_0 = (1 - R_0\sqrt{C/L})(1 + R_0\sqrt{C/L})^{-1}
\end{aligned} \tag{3}$$

which is a differential equation coupled with a difference equation in continuous time. A similar approach of applying formally the Laplace transform and deducing a characteristic equation accounting for time delays (deviating arguments was used in the pioneering papers [5]–[7] dealing with steam pipes; for water pipes a pioneering paper is [8] where the same approach is applied.

It is useful to continue the investigation of the above benchmark system by observing that the aggregate

$$v(t) + \sqrt{L/C}i(1,t) \equiv u(1,t) + \sqrt{L/C}i(1,t)$$

represents the so called progressive (forward) wave of the system at the boundary $\lambda = 1$. Since both the voltage $u(\lambda, t)$ and the current $i(\lambda, t)$ are linear combinations of the progressive (forward) and reflected (backward) waves as follows

$$\begin{aligned}
u(\lambda,t) &= \frac{1}{2}[u_1(\lambda,t) + u_2(\lambda,t)] \\
i(\lambda,t) &= \frac{1}{2}\sqrt{C/L}[u_1(\lambda,t) - u_2(\lambda,t)]
\end{aligned} \tag{4}$$

it is useful to express (2) in terms of these waves

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial u_1}{\partial \lambda} = 0\ ,\ \frac{\partial u_2}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial u_2}{\partial \lambda} = 0 \\
&(1 + R_0\sqrt{C/L})u_1(0,t) + (1 - R_0\sqrt{C/L})u_2(0,t) = \\
&= 2E(t)\ ;\ u_1(1,t) + u_2(1,t) = 2v(t) \\
&C_0\frac{\mathrm{d}v}{\mathrm{d}t} + \psi(v) = \frac{1}{2}\sqrt{C/L}[u_1(\lambda,t) - u_2(\lambda,t)]
\end{aligned} \tag{5}$$

It is obvious that the propagation (partial differential) equations of the two waves are decoupled; the two waves are exactly the Riemann invariants of the problem. We may consider now the standard version of the d’Alembert method i.e. of integrating along the two families of characteristics

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \pm\sqrt{LC} \tag{6}$$

Worth mentioning that there is a family of increasing characteristics and one of decreasing; as (5) shows, the forward wave should be considered along the increasing characteristics while the backward wave along the decreasing ones. If we perform this integration we shall find

$$u_1(0,t) = u_1(1, t + \sqrt{LC})\ ,\ u_2(1,t) = u_2(0, t + \sqrt{LC}) \tag{7}$$

By denoting

$$\eta_1(t) = u_1(1,t)\ ,\ \eta_2(t) = u_1(1,t) \tag{8}$$

we associate to (2) the following system of equations with delayed argument

$$\begin{aligned}
&C_0\frac{\mathrm{d}v}{\mathrm{d}t} + \psi(v) = \frac{1}{2}\sqrt{\frac{C}{L}}(\eta_1(t) - \eta_2(t - \sqrt{LC})) \\
&\eta_2(t) = -\rho_0\eta_1(t - \sqrt{LC}) + (1+\rho_0)E(t) \\
&\eta_1(t) = -\eta_2(t - \sqrt{LC}) + 2v(t)
\end{aligned} \tag{9}$$

which is exactly (3) but associated in a rigorous way, starting from the solutions of (2); even the initial conditions may be associated in this way. Moreover, the converse association is also possible. Using the representation formulae for the two waves

$$\begin{aligned}
&u_1(\lambda,t) = \eta_1(t + (1-\lambda)\sqrt{LC}) \\
&u_2(\lambda,t) = \eta_2(t + \lambda\sqrt{LC})
\end{aligned} \tag{10}$$

we may construct the solutions of (2) starting from the solutions of (9).

To end this introductory discussion we just mention that (7) and (10) define what is usually known as *lossless propagation*. Since the two waves propagate from one boundary to the other in finite time $\tau = \sqrt{LC}$ but without changing their waveform (just with a pure - lumped - time delay) this propagation is also distortionless. In the following we shall discuss both these aspects.

## II. Lossless and distortionless propagation

**A.** We shall consider again the Nagumo-Shimura circuit but with a lossy transmission line

$$\begin{aligned}
&\frac{\partial u}{\partial \lambda} + L\frac{\partial i}{\partial t} + Ri = 0\ ,\ \frac{\partial i}{\partial \lambda} + C\frac{\partial u}{\partial t} + Gu = 0 \\
&R_0 i(0,t) + u(0,t) = E(t)\ ,\ u_1(t) = v(t) \\
&C_0\frac{\mathrm{d}v}{\mathrm{d}t} + \psi(v) = i(1,t)
\end{aligned} \tag{11}$$

Introducing the forward and backward waves as previously i.e. using (4) we find

$$\begin{aligned}&\frac{\partial u_1}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial u_1}{\partial \lambda}+\frac{1}{2}(R/L+G/C)u_1+\\&\qquad+\frac{1}{2}(R/L-G/C)u_2=0\\&\frac{\partial u_2}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial u_2}{\partial \lambda}-\frac{1}{2}(R/L-G/C)u_1+\\&\qquad+\frac{1}{2}(R/L+G/C)u_2=0\end{aligned}\tag{12}$$

$$\begin{aligned}&(1+R_0\sqrt{C/L})u_1(0,t)+\\&\quad+(1-R_0\sqrt{C/L})u_2(0,t)=2E(t)\\&u_1(1,t)+u_2(1,t)=2v(t)\\&C_0\frac{\mathrm{d}v}{\mathrm{d}t}+\psi(v)=\frac{1}{2}\sqrt{\frac{C}{L}}[u_1(\lambda,t)-u_2(\lambda,t)]\end{aligned}$$

These equations are no longer decoupled unless the "matching" condition of Heaviside is met i.e. $RC = LG$ which "destroys" the coupling terms

$$\begin{aligned}&\frac{\partial u_1}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial u_1}{\partial \lambda}+(R/L)u_1=0\\&\frac{\partial u_2}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial u_2}{\partial \lambda}+(R/L)u_2=0\end{aligned}\tag{13}$$

We introduce the new "waves"

$$\begin{aligned}&u_1(\lambda,t)=\mathrm{e}^{-\delta\lambda}w_1(\lambda,t)\ ,\ u_2(\lambda,t)=\mathrm{e}^{\delta\lambda}w_2(\lambda,t)\\&\delta=R\sqrt{C/L}\end{aligned}\tag{14}$$

to obtain a lossless-like system

$$\begin{aligned}&\frac{\partial w_1}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial w_1}{\partial \lambda}=0\ ,\ \frac{\partial w_2}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial w_2}{\partial \lambda}=0\\&(1+R_0\sqrt{C/L})w_1(0,t)+(1-R_0\sqrt{C/L})w_2(0,t)=\\&=2E(t)\ ;\ \mathrm{e}^{-\delta}w_1(1,t)+\mathrm{e}^{\delta}w_2(1,t)=2v(t)\\&C_0\frac{\mathrm{d}v}{\mathrm{d}t}+\psi(v)=\frac{1}{2}\sqrt{\frac{C}{L}}[\mathrm{e}^{-\delta}w_1(1,t)-\mathrm{e}^{\delta}w_2(1,t)]\end{aligned}\tag{15}$$

Denoting now

$$\eta_1(t)=w_1(1,t)\ ,\ \eta_2(t)=w_2(0,t)\tag{16}$$

we associate to (15) the system

$$\begin{aligned}&C_0\frac{\mathrm{d}v}{\mathrm{d}t}+\psi(v)=\frac{1}{2}\sqrt{\frac{C}{L}}[\mathrm{e}^{-\delta}\eta_1(t)-\mathrm{e}^{\delta}\eta_2(t-\sqrt{LC})]\\&\eta_2(t)=-\rho_0\eta_1(t-\sqrt{LC})+(1+\rho_0)E(t)\\&\eta_1(t)=-\mathrm{e}^{-2\delta}\eta_2(t-\sqrt{LC})+2\mathrm{e}^{-\delta}v(t)\end{aligned}\tag{17}$$

and an additional damping is introduced in the second difference equation. Adapting (10) to the new case, we have

$$\begin{aligned}u_1(\lambda,t)&=\mathrm{e}^{-\delta\lambda}\eta_1(t+(1-\lambda)\sqrt{LC})\\u_2(\lambda,t)&=\mathrm{e}^{\delta\lambda}\eta_2(t+\lambda\sqrt{LC})\end{aligned}\tag{18}$$

and it is easily seen that the progressive wave propagates forwards from $\lambda = 0$ to $\lambda = 1$ being retarded and damped along the propagation while the reflected wave propagates backwards from $\lambda = 1$ to $\lambda = 0$ being also retarded and damped. Since the basic waveforms $\eta_i(\cdot)$ are not modified but just retarded during propagation, the propagation is also distortionless.

**B.** The natural development of the distortionless propagation is to consider the so called inhomogeneous media and transmission lines. The theory of the waveguides is their most straightforward application. The mathematical model of the inhomogeneous transmission line is given by the space varying telegraph equations [9]

$$\begin{aligned}-\frac{\partial v}{\partial \lambda}&=r(\lambda)i(\lambda,t)+l(\lambda)\frac{\partial i}{\partial t}\\-\frac{\partial i}{\partial \lambda}&=g(\lambda)v(\lambda,t)+c(\lambda)\frac{\partial v}{\partial t}\end{aligned}\tag{19}$$

with the standard notations, the line having length $L$. Here $l(\lambda) > 0$, $c(\lambda) > 0$ for standard physical reasons. The distortionless definition (*op. cit.*) states that

$$v(\lambda,t)=f(\lambda)\phi(t-\tau(\lambda))\tag{20}$$

where $f(\cdot)$ is called *attenuation* and $\tau(\cdot)$ is called *propagation delay* while $\phi(\cdot)$ is the waveform. Two are here the remarks to be made: i) only the progressive wave is considered i.e. $\tau(\lambda) > 0$; ii) only the voltage wave is concerned in this basic definition (*op. cit.*). The first aspect means the absence of the reflected wave; consequently the line is closed on an impedance $Z(L)$ that equals the characteristic impedance of the line. Mathematically speaking, this is a boundary condition at $\lambda = L$. There exist also other cases of interest, for instance the time independent voltage/curent ratio i.e. when $v(\lambda, t)$ is given by (20) and

$$i(\lambda,t)=h(\lambda)\phi(t-\tau(\lambda))\tag{21}$$

i.e. when the line is resistive. Our approach includes these cases in the general setting of the distortionless propagation. Since (19) are exactly like (11), we introduce the Riemann invariants by

$$u^{\pm}(\lambda,t)=v(\lambda,t)\pm a(\lambda)i(\lambda,t)\tag{22}$$

or by the converse equalities

$$\begin{aligned}v(\lambda,t)&=\frac{1}{2}[u^{+}(\lambda,t)+u^{-}(\lambda,t)]\\i(\lambda,t)&=\frac{1}{2a(\lambda)}[u^{+}(\lambda,t)-u^{-}(\lambda,t)]\end{aligned}\tag{23}$$

With the choice $a(\lambda) = \sqrt{l(\lambda)/c(\lambda)}$ which is similar to (14) the cross derivative terms are "destroyed" and the following equations are obtained

$$\begin{aligned} -\frac{\partial u^+}{\partial \lambda} &= \sqrt{l(\lambda)c(\lambda)}\,\frac{\partial u^+}{\partial t} + \\ &+ \frac{1}{2}\left(a(\lambda)g(\lambda) + \frac{r(\lambda)-a'(\lambda)}{a(\lambda)}\right)u^+(\lambda,t) + \\ &+ \frac{1}{2}\left(a(\lambda)g(\lambda) - \frac{r(\lambda)-a'(\lambda)}{a(\lambda)}\right)u^-(\lambda,t) \\ -\frac{\partial u^-}{\partial \lambda} &= -\sqrt{l(\lambda)c(\lambda)}\,\frac{\partial u^-}{\partial t} - \\ &- \frac{1}{2}\left(a(\lambda)g(\lambda) - \frac{r(\lambda)+a'(\lambda)}{a(\lambda)}\right)u^+(\lambda,t) - \\ &- \frac{1}{2}\left(a(\lambda)g(\lambda) + \frac{r(\lambda)+a'(\lambda)}{a(\lambda)}\right)u^-(\lambda,t) \end{aligned} \tag{24}$$

It is now rather obvious that the off-diagonal terms cannot be canceled by the same choice of the line coefficients. This explains the option in [9] for the distortionless propagation forwards: such choice requires decoupling of the equation of $u_1(\lambda,t)$ in (24). Therefore

$$a'(\lambda) = r(\lambda) - g(\lambda)a^2(\lambda) \tag{25}$$

which is a condition on line's parameters. Remark that *this is a Riccati differential equation*. Consequently the equations of the waves become

$$\begin{aligned} -\frac{\partial u^+}{\partial \lambda} &= \sqrt{l(\lambda)c(\lambda)}\frac{\partial u^+}{\partial t} + a(\lambda)g(\lambda)u^+(\lambda,t) \\ -\frac{\partial u^-}{\partial \lambda} &= -\sqrt{l(\lambda)c(\lambda)}\frac{\partial u^-}{\partial t} - \frac{r(\lambda)}{a(\lambda)}u^-(\lambda,t) + \\ &+ (r(\lambda)/a(\lambda) - g(\lambda)a(\lambda))u^+(\lambda,t) \end{aligned} \tag{26}$$

Having in mind (14) we introduce the new "waves" by

$$\begin{aligned} u^+(\lambda,t) &= \exp\left(-\int_0^\lambda g(\sigma)a(\sigma)\mathrm{d}\sigma\right) w^+(\lambda,t) \\ u^-(\lambda,t) &= \exp\left(-\int_\lambda^1 (r(\sigma)/a(\sigma)\mathrm{d}\sigma\right) w^-(\lambda,t) \end{aligned}$$

to obtain

$$\begin{aligned} -\frac{\partial w^+}{\partial \lambda} &= \sqrt{l(\lambda)c(\lambda)}\frac{\partial w^+}{\partial t} \\ -\frac{\partial w^-}{\partial \lambda} &= -\sqrt{l(\lambda)c(\lambda)}\frac{\partial w^-}{\partial t} - \beta(\lambda)w^+(\lambda,t) \end{aligned} \tag{27}$$

where

$$\begin{aligned} \beta(\lambda) &= (-r(\lambda)/a(\lambda) + g(\lambda)a(\lambda)) \times \\ &\times \exp\left(-\int_0^\lambda g(\sigma)a(\sigma)\mathrm{d}\sigma + \int_\lambda^1 (r(\sigma)/a(\sigma)\mathrm{d}\sigma\right) \end{aligned} \tag{28}$$

We perform now integration along the characteristics to find

$$w^+(0,t) = w^+(L, t+\tau) \tag{29}$$

Denoting $\eta^+(t) = w^+(L,t)$ the following representation formula is obtained

$$w^+(\lambda,t) = \eta^+\left(t + \int_\lambda^L \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu\right) \tag{30}$$

obviously accounting for distortionless propagation of the forward wave. For the backward wave we obtain, by integrating along the decreasing characteristics but taking also into account (30)

$$\begin{aligned} w^-(L,t) &= w^-(0,t+\tau) + \\ &+ \int_0^L \beta(\sigma)\eta^+\left(t + 2\int_\sigma^L \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu\right)\mathrm{d}\sigma \end{aligned} \tag{31}$$

Denoting $\eta^-(t) = w^-(0,t+\tau)$ the following representation formula is obtained

$$\begin{aligned} w^-(\lambda,t) = \eta^-&\left(t + \int_0^\lambda \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu\right) + \\ +\int_0^\lambda \beta(\sigma)\eta^+&\left(t + \int_\sigma^\lambda \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu + \right. \\ &\left. + \int_\sigma^L \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu\right)\mathrm{d}\sigma \end{aligned} \tag{32}$$

and the propagation is clearly associated with the distortions introduced by the integral term. To obtain distortionless of the backward wave, it is necessary to have $\beta(\sigma) = 0$ a.e. that is $a(\lambda)g(\lambda) = r(\lambda)/a(\lambda)$ If $a(\lambda)$ is replaced by its expression i.e. $a(\lambda) = \sqrt{l(\lambda)/c(\lambda)}$ we obtain

$$\begin{aligned} g(\lambda)\sqrt{l(\lambda)/c(\lambda)} &= r(\lambda)\sqrt{c(\lambda)/l(\lambda)} \\ &\Updownarrow \\ g(\lambda)l(\lambda) &= r(\lambda)c(\lambda) \end{aligned} \tag{33}$$

which is exactly the Heaviside condition. However this condition is valid only for those $a(\lambda)$ satisfying (25). This gives $a'(\lambda) = 0$ hence the ratio $l(\lambda)/c(\lambda)$ *has to be piecewise constant on* $(0, L)$. *Not only constant coefficients can ensure distortionless propagation for both forward and backward waves!*

## III. The multi-wave case. Application to the circulating fuel nuclear reactors

When several transmission lines (channels) are included in the system, several couples of waves are present, leading to the model of e.g. [10]

$$\frac{\partial u}{\partial t}+A(\lambda)\frac{\partial u}{\partial \lambda}=B(\lambda)u\ ,\ t>0\ ,\ 0\leq\lambda\leq L \tag{34}$$

where $u$ is a $m$-dimensional vector and $A(\lambda)$, $B(\lambda)$ are $m\times m$ matrices. Also $A$ is supposed diagonal, having distinct diagonal elements, of which $k$ are strictly positive (corresponding to the forward waves) and $m-k$ are strictly negative (corresponding to the backward waves). If $B(\lambda)$ could be also diagonal then propagation would be distortionless, otherwise it is not.

The structure described by (34) arises from a more general problem. Consider a system which is symmetric in the sense of Friedrichs which looks like (34) but with $A(\lambda)$ only symmetric (but with distinct non-zero eigenvalues). Usually a nonsingular change of function is considered to diagonalize this matrix. More specific, if we consider the system

$$\frac{\partial v}{\partial t}+C(\lambda)\frac{\partial v}{\partial \lambda}+D(\lambda)v=0, \tag{35}$$

let $T(\lambda)$ be a nonsingular matrix such that $T^{-1}(\lambda)C(\lambda)T(\lambda)$ is diagonal. We take $u(\lambda,t)=T(\lambda)v(\lambda,t)$ to find

$$\begin{aligned}T(\lambda)\frac{\partial u}{\partial t}+C(\lambda)\left(T'(\lambda)u+T(\lambda)\frac{\partial u}{\partial \lambda}\right)+\\ +D(\lambda)T(\lambda)u=0\end{aligned} \tag{36}$$

from where the form (34) is obtained with $A(\lambda)=T^{-1}(\lambda)C(\lambda)T(\lambda)$ diagonal and

$$B(\lambda)=T^{-1}(\lambda)(C(\lambda)T'(\lambda)+D(\lambda)T(\lambda))$$

It appears that distortionless could be achieved in very special cases where $B(\lambda)$ defined above would result also diagonal.

There exist however situations when this diagonal structure is inherent to the basic equations. This, for instance, the case of the circulating fuel nuclear reactor: we deal here with a model of [11]–[13]

$$\begin{cases}\dfrac{d}{dt}n(t)=\rho n(t)+\displaystyle\sum_{i=1}^{m}\beta_i(\bar{c}_i(t)-n(t)),\\ \bar{c}_i(t)=\displaystyle\int_0^h\phi(\eta)c_i(\eta,t)d\eta,\quad i=\overline{1,m}\\ \dfrac{\partial c_i}{\partial t}+\dfrac{\partial c_i}{\partial \eta}+\sigma_i c_i=\sigma_i\phi(\eta)n(t),\\ c_i(0,t)=c_i(h,t),\quad i=\overline{1,m},\quad t\geq t_0\\ c_i(\eta,t_0)=q_i^0(\eta),\quad n(t_0)=n_0,\quad 0\leq\eta\leq h.\end{cases} \tag{37}$$

This model contains the delayed neutron equations accounting for the hydrodynamic equations of the circulating fuel. We gave here also the boundary conditions which are of periodic type. The PDE (partial differential equations) are completely decoupled, the coupling taking place at the level of the differential equation which is controlled by the average values taken from the PDE's but is itself controlling the PDE's in a distributed way. Also all eigenvalues of $A(\lambda)=I$ are equal and positive hence there exist $m$ forward waves. Integration along the characteristics and computation of the integral of (37) - which is quite involved - gives the following system of functional differential equations

$$\begin{cases}\dfrac{dn}{dt}=(\rho-\displaystyle\sum_{1}^{m}\beta_i)n(t)+\\ +\displaystyle\sum_{i=1}^{m}\beta_i\sigma_i\int_{-h}^{0}e^{\lambda\sigma_i}\left(\int_{-\lambda}^{h}\phi(\lambda)\phi(\eta+\lambda)d\eta\right)n(t+\lambda)d\lambda\\ \quad+\displaystyle\sum_{i=1}^{m}\beta_i\int_{-h}^{0}e^{\eta\sigma_i}\phi(-\eta)q_i(t+\eta)d\eta\\ q_i(t+h)=e^{-h\sigma_i}\left[q_i(t)+\sigma_i\displaystyle\int_0^h e^{\lambda\sigma_i}\phi(\lambda)n(t+\lambda)d\lambda\right].\end{cases} \tag{38}$$

with a corresponding system of initial conditions. Observe that the equations for $q_i$ contain a lumped time delay accounting for distortionless propagation while the variable $n$ displays a distributed delay due to the way it enters the PDE's. The difference equations for $q_i(t)$ show a system of FDE (functional differential equations) of neutral type. Moreover, if the representation formula is used

$$\begin{aligned}c_i(\eta,t)\ =\ &\mathrm{e}^{(h-\eta)\sigma_i}[q_i(t+h-\eta)-\\ &-\ \sigma_i\int_{-h+\eta}^{0}\mathrm{e}^{\theta\sigma_i}\varphi(\theta)n(t+h+\theta-\eta)\mathrm{d}\theta]\end{aligned} \tag{39}$$

it may be observed that the propagation is distortionless according to the definition given in the previous section.

## IV. Two control problems

We have selected here two stabilization problems for engineering systems containing elastic rods. What makes the difference is that in the first case lossless propagation is present while in the second the nonhomogeneous material properties account for propagation with distortions.

**A.** The controlled flexible arm of unitary mass and length, with the control at the boundary $\lambda=0$ is described by [14]

$$\begin{aligned}&\frac{\partial^2 y}{\partial t^2}-\frac{\partial^2 y}{\partial \lambda^2}=0\ ,\ 0\leq\lambda\leq1\ ,\ t\geq0\\ &\frac{\partial y}{\partial\lambda}(0,t)=\tau(t)\ ,\ \frac{\partial y}{\partial\lambda}(1,t)=0\end{aligned} \tag{40}$$

where $y(\lambda,t)$ is the torsion angle and $\tau(t)$ is the boundary control, supposed to have the form

$$\begin{aligned}\tau(t)=-k_p\left[y(0,t)-\int_0^1k(\lambda)\frac{\partial y}{\partial\lambda}(\lambda,t)\mathrm{d}\lambda\right]-\\ -k_v\left[\frac{\partial y}{\partial\lambda}(0,t)-\int_0^1k(\lambda)\frac{\partial^2 y}{\partial\lambda\partial t}(\lambda,t)\mathrm{d}\lambda\right]\end{aligned} \tag{41}$$

At the same time the model for stabilizing the vibrations of a torsion beam by boundary control [15] uses the same equations, the same control law (41) (regardless an integration by parts) but applied at the boundary $\lambda = 1$

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial \lambda^2} = 0\ ,\ 0 \leq \lambda \leq 1\ ,\ t \geq 0 \\ &\frac{\partial y}{\partial \lambda}(0,t) = 0\ ,\ \frac{\partial y}{\partial \lambda}(1,t) = \tau(t) \end{aligned} \tag{42}$$

We shall discuss these two models - in fact a single one - as follows. Introduce first the new functions

$$v(\lambda,t) = \frac{\partial y}{\partial t}(\lambda,t)\ ,\ w(\lambda,t) = \frac{\partial y}{\partial \lambda}(\lambda,t) \tag{43}$$

to obtain the standard equations

$$\begin{aligned} &\frac{\partial v}{\partial t} = \frac{\partial w}{\partial \lambda}\ ,\ \frac{\partial w}{\partial t} = \frac{\partial v}{\partial \lambda} \\ &w(0,t) = \tau(t)\ ,\ w(1,t) = 0 \end{aligned} \tag{44}$$

and, after an integration by parts

$$\begin{aligned} \tau(t) = &-k_p\left[y(1,t) - \int_0^1 (1+k(\lambda)w(\lambda,t)\mathrm{d}\lambda\right] - \\ &-k_v[(1+k(0))v(0,t) - k(1)v(1,t)] - \\ &-k_v\int_0^1 (k(\lambda)+k'(\lambda))v(\lambda,t)\mathrm{d}\lambda \end{aligned} \tag{45}$$

We may now introduce the forward and backward waves

$$\begin{aligned} &u^{\pm}(\lambda,t) = v(\lambda,t) \mp w(\lambda,t) \\ &v(\lambda,t) = \tfrac{1}{2}[(u^-(\lambda,t) + u^+(\lambda,t)] \\ &w(\lambda,t) = \tfrac{1}{2}[(u^-(\lambda,t) - u^+(\lambda,t)] \end{aligned} \tag{46}$$

to obtain the equations of the lossless propagation

$$\begin{aligned} &\frac{\partial u^+}{\partial t} = -\frac{\partial u^+}{\partial \lambda}\ ,\ \frac{\partial u^-}{\partial t} = \frac{\partial u^-}{\partial \lambda} \\ &u^-(0,t) - u^+(0,t) = 2\tau(t)\ ,\ u^-(1,t) - u^+(t) = 0 \end{aligned} \tag{47}$$

and the control re-written

$$\begin{aligned} \tau(t) = &-k_p\left[y(1,t) - \frac{1}{2}\int_0^1 (1+k(\lambda)(u^-(\lambda,t) - \right. \\ &\qquad\qquad -u^+(\lambda,t))\ \mathrm{d}\lambda] - \\ &-\tfrac{1}{2}k_v(1+k(0))(u^-(0,t) + u^+(0,t)) + \\ &+\tfrac{1}{2}k_v k(1)(u^-(1,t) + u^+(1,t)) - \\ &-\tfrac{1}{2}k_v\int_0^1 (k(\lambda)+k'(\lambda))(u^-(\lambda,t) + u^+(\lambda,t))\mathrm{d}\lambda \end{aligned} \tag{48}$$

We left aside till now the term $y(1,t)$ - the angle at the control boundary - it might be considered as some dynamical set point and, due to linearity, we take it identically 0 from now on. We continue by integrating along the characteristics; adapting (7) and (8) we find

$$u^+(0,t) = u^+(1,t+1)\ ,\ u^-(1,t) = u^-(0,t+1) \tag{49}$$

and may define

$$\begin{aligned} &\eta^+(t) = u^+(1,t)\ ,\ \eta^-(t) = u^-(0,t) \\ &u^+(\lambda,t) = \eta^+(t+1-\lambda)\ ,\ u^-(\lambda,t) = \eta^-(t-\lambda) \end{aligned} \tag{50}$$

This allows association of the functional equations - deduced by substituting (49), (50) in the boundary conditions of (47)

$$\begin{aligned} &\eta^-(t) - \eta^+(t+1) = 2\tau(t)\ ,\ \eta^+(t) = \eta^-(t+1) \\ &2\tau(t) = k_p\int_0^1 (1+k(\lambda)(\eta^-(t-\lambda) - \eta^+(t+1-\lambda))\mathrm{d}\lambda - \\ &-k_v(1+k(0))(\eta^-(t) - \eta^+(t+1)) + \\ &+k_v k(1)(\eta^-(t+1) + \eta^+(t)) - \\ &-k_v\int_0^1 (k(\lambda)+k'(\lambda))(\eta^-(t-\lambda) + \eta^+(t+1-\lambda))\mathrm{d}\lambda \end{aligned}$$

Introducing the translated functions $\varsigma^{\pm}(t) = \eta^{\pm}(t+1)$ we shall obtain

$$\begin{aligned} &\varsigma^+(t) - \varsigma^-(t-1) = -2\tau(t)\ ,\ \varsigma^-(t) = \varsigma^+(t-1) \\ &2\tau(t) = k_p\int_0^1 (1+k(\lambda)(\varsigma^-(t-1-\lambda) - \varsigma^+(t-\lambda))\mathrm{d}\lambda - \\ &-k_v(1+k(0))(\varsigma^-(t-1) - \varsigma^+(t)) + \\ &+k_v k(1)(\varsigma^-(t) + \varsigma^+(t-1)) - \\ &-k_v\int_0^1 (k(\lambda)+k'(\lambda))(\varsigma^-(t-1-\lambda) + \varsigma^+(t-\lambda))\mathrm{d}\lambda \end{aligned}$$

We may further eliminate $\varsigma^-(t)$ to obtain finally the following integro-functional equation

$$\begin{aligned} &\varsigma^+(t) + \alpha_1\varsigma^+(t-1) + \alpha_2\varsigma^+(t-2) = \\ &= \int_{-1}^0 [\beta_0(\lambda)\varsigma^+(t+\lambda) + \beta_1(\lambda)\varsigma^+(t-2+\lambda)]\mathrm{d}\lambda \end{aligned} \tag{51}$$

This is a quite standard linear difference equation. Its stability may be studied *via* the characteristic equation

$$1 + \alpha_1 \mathrm{e}^{-s} + \alpha_2 \mathrm{e}^{-2s} = \int_{-1}^0 [\beta_0(\lambda) + \mathrm{e}^{-2s}\beta_1(\lambda)]\mathrm{e}^{\lambda s}\mathrm{d}\lambda \tag{52}$$

or by associating to (51) a Liapunov functional suggested by the energy integral of (44)-(45).

**B.** The control model of an overhead crane with a flexible cable is given by [16]

$$\begin{aligned}&y_{tt}-(a(s)y_s)_s=0\ ,\ t>0\ ,\ 0<s<L\\&y_{tt}(0,t)=gy_s(0,t)\ ,\ y(L,t)=X_p(t)\\&\ddot{X}_p=K(a(s)y(s,t))(L,t)+u(t)\\&a(s)=g(s+(m/\rho))\ ,\ K=\frac{m+\rho L}{ma(L)}=\frac{\rho}{Mg}\end{aligned}\tag{53}$$

in fact the starting model of (*op.cit.*) contained the boundary condition $y_s(0,t)=0$, explained by the physical assumption that the acceleration of the load mass is negligible with respect to the gravitational acceleration $g$ i.e. $y_{tt}(0,t)/g\approx 0$; in fact this is not rigorous and definitely cannot be ascertained for all $t$; the only valid argument is connected to singular perturbations. For this reason we shall deal with the complete model (53).

If the rated cable length variable $\sigma=s/L$ is introduced, then, with a slight abuse of notation, the following model containing possible small parameters is obtained

$$\begin{aligned}&\frac{L}{g}\cdot\frac{\rho L}{m}y_{tt}-\left(\left(1+\frac{\rho L}{m}\sigma\right)y_\sigma\right)=0\ ,\ 0\leq\sigma\leq 1\ ,\ t>0\\&\frac{L}{g}y_{tt}(0,t)=y_\sigma(0,t)\ ,\ y(1,t)=X_p(t)\\&\frac{L}{g}\ddot{X}_p=\frac{m}{M}\left(1+\frac{\rho L}{M}\right)y_\sigma(1,t)+\frac{L}{g}u(t)\end{aligned}\tag{54}$$

A preliminary comment is useful: supposing we would like to neglect non-uniformity of the cable parameters, this would require the assumption that the cable mass is negligible with respect to the carried mass i.e. $\rho L/m\approx 0$. However, this will destroy the entire distributed dynamics since (54) would become

$$\begin{aligned}&y_{\sigma\sigma}=0\ ;\ \frac{L}{g}y_{tt}(0,t)=y_\sigma(0,t)\ ,\ y(1,t)=X_p\\&\frac{L}{g}\ddot{X}_p=\frac{m}{M}y_\sigma(1,t)+\frac{L}{g}u(t)\end{aligned}\tag{55}$$

We shall then have $y(\sigma,t)=\phi_1(t)\sigma+\phi_0(t)$ which is substituted in the boundary conditions. Therefore

$$\begin{aligned}&\frac{L}{g}\ddot{\phi}_0+\phi_0=X_p\ ;\ \phi_1=X_p-\phi_0\\&\frac{L}{g}\ddot{X}_p=\frac{m}{M}(X_p-\phi_0)+\frac{L}{g}u(t)\end{aligned}\tag{56}$$

Its uncontrolled dynamics is given by the roots of the characteristic equation

$$\frac{L}{g}s^2\left(\frac{L}{g}s^2+1-\frac{m}{M}\right)=0\tag{57}$$

i.e. by two purely imaginary modes and a double zero mode; this is but well known. Instead of this approach, we start by introducing new functions and by making some other notations

$$\begin{aligned}&v(\sigma,t):=y_t(\sigma,t)\ ,\ w(\sigma,t):=(1+\gamma_0\sigma)y_\sigma(\sigma,t)\\&\gamma_0=\frac{\rho L}{m}\ ,\ T^2=\frac{L}{g}\ ,\ \delta_0=\frac{m}{M}\end{aligned}\tag{58}$$

thus obtaining

$$\begin{aligned}&\gamma_0T^2v_t=w_\sigma\ ,\ w_t=(1+\gamma_0\sigma)v_\sigma\\&T^2v_t(0,t)=w(0,t)\ ,\ v(1,t)=\dot{X}_p\\&T^2\ddot{X}_p=\delta_0w(1,t)+T^2u(t)\end{aligned}\tag{59}$$

Define further the forward and backward waves as below

$$\begin{aligned}v(\sigma,t)&=u^+(\sigma,t)+u^-(\sigma,t)\\w(\sigma,t)&=T\sqrt{\gamma_0(1+\gamma_0\sigma)}(u^-(\sigma,t)-u^+(\sigma,t))\\u^+(\sigma,t)&=\frac{1}{2}\left(v(\sigma,t)-\frac{1}{T\sqrt{\gamma_0(1+\gamma_0\sigma)}}w(\sigma,t)\right)\\u^-(\sigma,t)&=\frac{1}{2}\left(v(\sigma,t)+\frac{1}{T\sqrt{\gamma_0(1+\gamma_0\sigma)}}w(\sigma,t)\right)\end{aligned}\tag{60}$$

The following equations are then obtained

$$\begin{aligned}&\frac{\partial u^+}{\partial t}+\frac{1}{T\sqrt{\gamma_0}}\sqrt{(1+\gamma_0\sigma)}\frac{\partial u^+}{\partial t}=\\&=\frac{1}{T\sqrt{\gamma_0}}\cdot\frac{\gamma_0}{\sqrt{(1+\gamma_0\sigma)}}(u^--u^+)\\&\frac{\partial u^-}{\partial t}-\frac{1}{T\sqrt{\gamma_0}}\sqrt{(1+\gamma_0\sigma)}\frac{\partial u^-}{\partial t}=\\&=\frac{1}{T\sqrt{\gamma_0}}\cdot\frac{\gamma_0}{\sqrt{(1+\gamma_0\sigma)}}(u^--u^+)\\&T(u_t^-+u_t^+)(0,t)=\sqrt{\gamma_0}(u^-(0,t)-u^+(0,t))\\&u^-(1,t)+u^+(1,t)=\dot{X}_p\\&T\ddot{X}_p=\delta_0\sqrt{\gamma_0(1+\gamma_0)}(u^-(1,t)-u^+(1,t))+Tu(t)\end{aligned}\tag{61}$$

It is clear that *under no conditions can be made this system distortionless*. We may however try to replace this system by an approximation which would be such. To find such an approximation, we turn back to the basic equation

$$y_{tt}-(a(s)y_s)_s=0$$

where $a(\cdot)$ is a sufficiently smooth function. With the new variables

$$v(s,t)=y_t(s,t)\ ,\ w(s,t)=a(s)y_s(s,t)$$

the first order equations of the propagation are obtained

$$v_t=w_s\ ,\ w_t=a(s)v_s$$

The forward and backward waves are defined by

$$v(s,t) = u^-(s,t) + u^+(s,t)$$
$$w(s,t) = \sqrt{a(s)}(u^-(s,t) - u^+(s,t))$$

and satisfy

$$\begin{aligned} u_t^+ + \sqrt{a(s)}u_s^+ &= \frac{a'(s)}{4\sqrt{a(s)}}(u^- - u^+) \\ u_t^- - \sqrt{a(s)}u_s^- &= \frac{a'(s)}{4\sqrt{a(s)}}(u^- - u^+) \end{aligned} \tag{62}$$

Obviously the distortionless condition is $a'(s) = 0$ a.e. Since in our case $a(s) = g(s+m/\rho)$, $a'(s) = g \neq 0$. The piecewise constant approximation is thus the only suitable. This means approximation of $a(s)$ piecewise constantly in order that e.g. the propagation time should remain constant

$$\int_0^L \frac{\mathrm{d}\lambda}{\sqrt{a(\lambda)}} = \sum_1^N \frac{l_i}{\sqrt{a_i}} \; , \; \sum_1^N l_i = L \tag{63}$$

We shall not discuss here specific approximation problems such as concatenation conditions and convergence but just take $N = 1$ and write down the associated system. In this simplest case we find

$$T\sqrt{\gamma_0}\int_0^1 \frac{\mathrm{d}\sigma}{\sqrt{1+\gamma_0\sigma}} = T\sqrt{\gamma_0}\frac{1}{\sqrt{1+\gamma_1}} \tag{64}$$

If $T_d = T\sqrt{\gamma_0/(1+\gamma_1)}$ - the propagation time - is introduced and the cyclic variable $X_p$ is eliminated, we obtain a genuine system of neutral type

$$\begin{aligned} &T_d\frac{\mathrm{d}}{\mathrm{d}t}(y^+(t) + y^-(t-T_d)) = -\gamma_0(y^+(t) - y^-(t-T_d)) \\ &T_d\frac{\mathrm{d}}{\mathrm{d}t}(y^-(t) + y^+(t-T_d)) = \gamma_0\delta_0(y^-(t) - y^+(t-T_d)) + \\ &\qquad\qquad + T_d u(t) \\ &\dot{X}_p = y^-(t) + y^+(t-T_d) \end{aligned} \tag{65}$$

where we denoted

$$\begin{aligned} y^+(t) &= u^+(0,t) \; , \; y^-(t) = u^-(1,t) \; , \\ v(\sigma,t) &= u^-(\sigma,t) + u^+(\sigma,t) \; , \\ w(\sigma,t) &= T\sqrt{\gamma_0(1+\gamma_1)}(u^-(\sigma,t) + u^+(\sigma,t)) \end{aligned}$$

For $u(t) \equiv 0$ the inherent stability of (65) has been studied [17]. Its characteristic equation

$$\begin{aligned} &(T_d s + \gamma_0)(T_d s - \gamma_0\delta_0) - \\ &\quad -(T_d s - \gamma_0)(T_d s + \gamma_0\delta_0)\mathrm{e}^{-2sT_d} = 0 \end{aligned} \tag{66}$$

and obviously has a zero root; if we take into account that the output $\dot{X}_p$ is a cyclic variable, we rediscover a fact known from the lumped parameter case - see (57) - a double zero root of the controlled configuration with $u(t)$ as input and $X_p$ as measurable control output. Since (57) has also a pair of of purely imaginary roots, we may check for purely imaginary roots of (66) and find them to be of the form $\pm\imath x_k/Td$ where $x_k$ are the positive roots of

$$\tan\, x = \frac{\gamma_0(1-\delta_0)x}{\gamma_0\delta_0 + x^2} \tag{67}$$

The equation is well studied [18]: it has real roots of the form $k\pi + \delta_k$ where $\{\delta_k\}_k$ is a positive bounded sequence approaching 0 for $k \to \infty$.

We have thus discovered an infinity of purely imaginary roots; this infinity of oscillating modes is well known in the theory of the elastic rods; mathematically, its presence can be explained by the fact that the difference operator of (65) has its roots on$\imath\mathbb{R}$ - the imaginary axis. Other details may be found in [17].

## V. Dynamics and control for systems of conservation laws

It is well known that the first applications in the field of control for systems with distributed parameters dealt with mainly with specific problems (e.g. pressure control in steam pipes, water hammer in hydraulics, as already mentioned in previous sections); however a more contemporary trend consists in applying control theory to general structures that may be considered as benchmark problems. Due to their broad applications, the systems of conservation laws which describe various physical phenomena with a single space parameter distribution are very suitable for such applications [19]. The systems of conservation laws are interesting also for their nonlinear character; when linearized they reduce to the quite well propagation equations - see [20] or the previous sections - and, therefore, a comparison to some known results is also available.

We shall consider in this section a system of two conservation laws on $\mathbb{R}^2$ (one space variable) which reads

$$Y_t + f(Y)_x = 0 \tag{68}$$

where $Y : [0,\infty) \times [0,L] \mapsto \Omega \subseteq \mathbb{R}^2$ is the vector of the two dependent variables and $f : \Omega \subseteq \mathbb{R}^2 \mapsto \mathbb{R}^2$ is the *flux density*. Unlike the mostly studied cases [19], the solution is defined by the initial conditions

$$Y(x,0) = Y_0(x) \; , \; 0 \leq x \leq L \tag{69}$$

and by some boundary conditions of Dirichlet type while they may nevertheless contain some control input variables

$$g_0(Y(0,t), u_0(t)) = 0 \; , \; g_L(Y(L,t), u_L(t)) = 0 \; , \; t > 0 \tag{70}$$

The standard problem we are approaching reads as follows

For constant control actions $u_i(t) \equiv \overline{u}_i \; , \; i = 0, L$, a steady state solution is a constant solution $\overline{Y}$ satisfying (68) and (70).

Depending on the form of the boundary conditions this steady state solution may be stable or unstable. Accordingly it may be stated the *boundary control problem* - that of defining the control inputs $u_i(t)$ from a feedback structure such that for any smooth enough initial condition in (69 the unique smooth solution should converge to a desired steady solution by $\overline{u}_i$ - the controllers' set points.

**A.** Consider the first application - the control of the flows in open canals [21]–[23] By choosing the flow velocity $V(x,t)$ and the cross section $A(x,t)$ (instead of the liquid level), the standard Saint Venant equations are expressed as conservation laws. We shall deal here with the simplest case of the *prismatic level canal* whose geometric parameters are independent of the coordinate $x$ and whose bed is lying at the same constant elevation $Y_b$. Under the circumstances the conservation laws are given by

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ V \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} AV \\ \frac{1}{2}V^2 + g\psi(A) \end{pmatrix} = 0 \qquad (71)$$

where $h = \psi(A)$ is the liquid level and $\psi$ is the inverse of the monotonic mapping defining the cross section is a prismatic canal

$$\Phi(h) = \int_0^h \sigma(y)\mathrm{d}y$$

($\sigma$ is the canal width corresponding to the liquid elevation $y$). To these equations we may add the boundary conditions which arise from the canal conditions; if we consider constant flow at $x = 0$ and constant level (area) at $x = L$ then

$$A(0,t)V(0,t) = Q_0 \ , \ A(L,t) = A_L \qquad (72)$$

Without reproducing the details, we give here an account of the results in [23] . Introducing the Riemann invariants - forward and backward waves - means diagonalizing of a matrix whose real eigenvalues are

$$\lambda^{\pm} = V \pm \sqrt{gA/\sigma\psi(A)} \qquad (73)$$

of which $\lambda^+$ is always strictly positive. *Hyperbolicity* of the system requires $\lambda^- < 0$ hence

$$\mathfrak{Fr}(A,V) = \frac{V}{\sqrt{gA/\sigma\psi(A)}} < 1 \qquad (74)$$

where $\mathfrak{Fr}(A,V)$ thus defined is the Froude number; condition (74) means that the flow is *fluvial* (subcritical).

Integration along the characteristics will send to the following functional equations to be satisfied by the boundary waves $y^+(t)$, $y^-(t)$

$$\begin{aligned} & y^+(t) = u^+(L,t) \quad , \quad y^-(0t,) = u^-(0,t) \\ & (y^+(t+T^+(t)) + y^-(t))F^{-1}(y^+(t+T^+(t)) - y^-(t)) = Q_0 \\ & y^+(t) - y^-(t+T^-(t)) = F(A_0) \end{aligned} \qquad (75)$$

where $T^{\pm}(t)$ are the propagation times on the characteristics while $F(A)$ is a monotone mapping defined by

$$F(A) = \int_0^A \sqrt{\frac{g}{\alpha\sigma(\psi(\alpha))}}\mathrm{d}\alpha \qquad (76)$$

Since there is a discussion whether it is better to consider the hyperbolic systems directly or *via* the associated functional equations obtained by integration along the characteristics, we mention here that *the use of the functional equations* gives better results than the direct method. In [23] the existence of the following physically significant invariant sets was proven

$$\begin{aligned} & -F(A_0) < V(x,t) < F(A_0) \quad , \quad 0 \leq x \leq L \ , \ t > 0 \\ & 0 < F(A(x,t)) < 2F(A_0) \quad , \quad 0 \leq x \leq L \ , \ t > 0 \end{aligned} \qquad (77)$$

This shows both limited flow reversals as well as some limitations of the liquid level. It is not quite clear if these conditions may ensure *the invariance of the Froude number* i.e. $\mathfrak{Fr}(A(x,t),V(x,t)) < 1$ provided $\mathfrak{Fr}(A(x,0),V(x,0)) < 1$ thus ensuring sub-criticality of the flow; actually one can hope that initial conditions that are sufficiently far away from the critical limit will generate sub-critical evolutions.

**B.** We shall address now to a problem that has been considered much earlier (see [20] but also its references). In the technology of combined heat electricity there are steam pipes whose dynamics affect the stability of the control systems for basic operating parameters. The traditional approach of pipe dynamics started from the equations of the hydrodynamic flow

$$\begin{aligned} & \frac{\partial w}{\partial t} + w\frac{\partial w}{\partial l} + \frac{1}{\rho}\frac{\partial p}{\partial l} = 0 \\ & \frac{\partial \rho}{\partial t} + \rho\frac{\partial w}{\partial l} + w\frac{\partial \rho}{\partial l} = 0 \end{aligned} \qquad (78)$$

where the flow characteristics (velocity $w$, mass density $\rho$ and steam pressure $p$) are also related by the polytropic equation

$$p/p_\infty = (\rho/\rho_\infty)^\kappa \qquad (79)$$

with the subscript $\infty$ accounting for steady state values and $\kappa > 1$ being the polytropic exponent.

The standard approach supposed neglecting the steam friction terms $w(\partial w/\partial l)$ and $w(\partial\rho/\partial l)$ assumed to be small and then to linearize the remaining partial differential equations. Further, there were linearized the boundary condition and the possible nonlinearity remained the sector restricted actuator characteristic. By introducing the rated (per cross section area) mass flow $\phi = \rho w$ and eliminating the pressure $p$ using the polytropic equation (79) we obtain a system of conservation laws

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \phi \end{pmatrix} + \frac{\partial}{\partial l}\begin{pmatrix} \phi \\ \phi^2/\rho + \gamma_\infty\rho^\kappa \end{pmatrix} = 0 \qquad (80)$$

where $\gamma_\infty = p_\infty(\rho_\infty)^{-\kappa}$ is the polytropic steady state constant. The boundary conditions are defined by the controlled admission of the steam into the pipe at $l = 0$ and the steam consumption from the pipe at $l = L$; remark that they are analogous to those of the previous application

$$\phi(0,t) = \phi_0(t) \quad , \quad \phi(L,t) = \sqrt{2}\,\gamma_\infty\,\frac{f(t)}{F}\,(\rho(L,t))^{\frac{\kappa+1}{2}} \tag{81}$$

Here $f(t)$ - the admission cross section of the steam to the consumer acts as a disturbance; since the steam has to be supplied at constant pressure, this pressure has to be the measured output. Taking into account that the controller acts using the control error, the controller equations might be as follows

$$\begin{aligned} &\psi_m^2\ddot{\zeta} + \psi_D\dot{\zeta} + \zeta = \gamma_\infty(\rho(0,t))^\kappa - p_\infty \\ &\dot{\eta} = -\varphi(\zeta + \gamma_0(\eta - \eta_\infty)) \end{aligned} \tag{82}$$

where $\eta_\infty$ - the steady state of the actuator - may be computed using the steady state equations of (80)-(80). Summarizing we obtained the following boundary value problem for a system of conservation laws, as follows

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \phi\end{pmatrix} + \frac{\partial}{\partial l}\begin{pmatrix}\phi\\ \phi^2/\rho + \gamma_\infty\rho^\kappa\end{pmatrix} = 0 \\ &\phi(0,t) = \phi_0(t)\ ,\ \phi(L,t) = \sqrt{2}\,\gamma_\infty\,\frac{f(t)}{F}\,(\rho(L,t))^{\frac{\kappa+1}{2}} \\ &\psi_m^2\ddot{\zeta} + \psi_D\dot{\zeta} + \zeta = \gamma_\infty(\rho(0,t))^\kappa - p_\infty \\ &\dot{\eta} = -\varphi(\zeta + \gamma_0(\eta - \eta_\infty)) \end{aligned} \tag{83}$$

The initial conditions defining the initial state are both with lumped parameters - $\zeta(0)$, $\dot{\zeta}(0)$, $\eta(0)$ - and with distributed parameters - $\phi(x,0)$ and $\rho(0)$.

For this system various problems may be stated, some of them being already mentioned at the previous application: discussion of the hyperbolicity, associated functional equations and basic theory, invariant sets, control synthesis, stability.

Taking into account the most recent references, the control of the systems of conservation laws is at its beginnings (at least in the nonlinear case). Our point of view is that *the most suitable approach would be to use the energy integral as a Liapunov functional* (possibly for synthesis purposes also). In this way a nonlinear counterpart of the standard results e.g. [20] may be obtained; in fact the previous results also dealt with nonlinearity but only in the boundary conditions while here even the partial differential equations are nonlinear. Finding the associated functional equations is still a challenge [23].

## VI. Conclusions

**A.** This survey is an attempt to discuss some dynamical models in automatic control that are connected with distributed parameters in one dimension. These models are described by boundary value problems for hyperbolic partial differential equations. We considered here the functional equations associated to these problems using the integration of the Riemann invariants along the characteristics. In the case of lossless and/or distortionless propagation these equations have deviations of the arguments that are constants. Such quite known models correspond to linear partial differential equations with possible nonlinear boundary conditions. The most interesting and significant fact is that these equations arise from the linearization of the equations of the conservation laws. The control of the nonlinear systems of conservation laws is one of the most recent challenges in engineering. Here the approach is based on Liapunov quadratic functionals induced by the linearized models; it is felt however that the energy integral combined with integration along the characteristics (which, in several cases, as pointed out in [19], are straight lines) could produce new advancement. And, last but not least, the so called *model validation* (basic theory, invariant sets) may turn helpful for better control issues.

**B.** The present paper emerged from the ideas, models and problems contained in an invited plenary exposition at the 6th 2009 IEEE International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE 2009), held in November 1013, 2009 in Toluca, Mexico. Its text was not published in Conference Proceedings, but a revised shortened form was published in a book of contributions [24]. Since then many developments were acknowledged on each of the directions listed in this paper. Citing even important references would at least double their list. The hopes go towards the interested readers.

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