On the Nonlinear Observability of Polynomial Dynamical Systems

Daniel Gerbet
Institute of Control Theory
Technische Universität Dresden
Dresden, Germany
daniel.gerbet1@tu-dresden.de

Klaus Röbenack
Institute of Control Theory
Technische Universität Dresden
Dresden, Germany
klaus.roebenack@tu-dresden.de

***Abstract*—Controllability and observability are important system properties in control theory. These properties cannot be easily checked for general nonlinear systems. This paper addresses the local and global observability as well as the decomposition with respect to observability of polynomial dynamical systems embedded in a higher-dimensional state-space. These criteria are applied on some example system.**

***Index Terms*—Nonlinear systems, observability, ideals, varieties**

I. INTRODUCTION

In control theory the observability of a system is an important property. There exist well known criteria to decide this property for linear systems, see [10], [12]. For nonlinear systems the concept of observability has been introduced in [11] based on the indistinguishability of states by means of their output trajectory. In contrast to linear systems this indistinguishability may occur only locally in some neighborhood of a state, or only for a subset of states.

Currently, no observability criterion for a general nonlinear state-space system is known. In [19], [20] an interval arithmetics approach has been used, which investigates a compact subset of the state-space. For polynomial dynamically systems a sufficient criterion for locally observability, following a different definition as used within this article, has been given in [2]. A global criterion, also sufficient, was stated in [28], and in [24], [25] using a quantifier elimination approach. Sufficient and necessary observability criteria based on algebraic geometry are found in [13]–[15]. The paper [3] formulates a sufficient and necessary global algebraic criterion.

This article is based on the conference paper [7] and the journal paper [8]. Based on the local observability criterion derived therein an attempt to decompose systems with respect to observability is made. Due to the considered system class this is only possible if the system is at no point locally observable. While this algebraic approach can avoid unnecessary singularities, the resulting (locally) observable subsystem will possess not locally observable points that are inherent to the original system. These points can nonetheless be computed using the same local observability criterion.

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This article is structures as follows: First, some mathematical preliminaries regarding ordinary differential equations and polynomial ideals as well as the observability concept from [11] are recalled in Section II. On this basis a sufficient and necessary criterion for global and local observability is given in Section III. It follows a discussion of the not observable case and the decomposition with respect to observability. Section IV is dedicated to some example systems, on which the criterion is applied. Finally, some conclusions are drawn in Section V.

II. PRELIMINARIES

A. Differential Equations

Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a connected, real-analytic manifold equal to the (real) zero set of the polynomials $g_1, \ldots, g_m$ with $g_i \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$. On this manifold consider a vector field $f : \mathcal{M} \to \mathrm{T}\mathcal{M}$ and the corresponding differential equation

$$\dot{x} = f(x), \quad x(0) \in \mathcal{M} \tag{1a}$$

as well as the output map

$$y = h(x) \tag{1b}$$

with an analytic function $h : \mathcal{M} \to \mathbb{R}^p$. The vectorial components of the fields f and h in the standard basis are considered to be polynomials in the ring $\mathbb{R}[x]$, too. These constraints ensure that the solution of (1a) exists for every initial condition at least locally by the Picard-Lindelöf theorem [1]. The *flow* $\varphi_t(\cdot)$ maps each initial value to its solution $x(t)$ at time t.

B. Lie Derivatives and Lie Series

For any real smooth scalar field $h : \mathcal{M} \to \mathbb{R}$ the *Lie derivative* of h along the vector field f is the directional derivative [17]

$$\mathrm{L}_f h(x) = \tfrac{\mathrm{d}}{\mathrm{d}t} h(\varphi_t(x))\big|_{t=0} = h'(x) f(x), \tag{2}$$

where h' denotes the gradient of h. Higher order Lie derivatives can be defined recursively by

$$\mathrm{L}_f^{k+1} h(x) = \mathrm{L}_f \mathrm{L}_f^k h(x), \qquad \mathrm{L}_f^0 h(x) = h(x). \tag{3}$$

The definition (2) can also be applied for a vector-valued function $h : \mathcal{M} \to \mathbb{R}^p$, where h' is then the Jacobian matrix

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of h. This generalization can be interpreted as component-wise computed Lie derivative

$$L_f h(x) = \begin{pmatrix} L_f h_1(x) \\ \vdots \\ L_f h_p(x) \end{pmatrix}. \quad (4)$$

This notation is quite common in nonlinear multivariable control [16]. Note that the Lie derivative (4) of a vector-values map must not be confused with the Lie derivative of vector fields, defined in a different way [17], [22].

Using the Lie derivative, the output trajectory $y(t)$ of (1) can be expanded into a convergent Mac-Laurin series

$$y(t) = \sum_{k=0}^{\infty} \tfrac{t^k}{k!} \, \mathrm{L}_f^k h(x(0)) \quad (5)$$

called a *Lie series* [21].

C. Distinguishability and Observability

Two states $x, z \in \mathcal{M}$ are called *indistinguishable on the interval* $[0, T]$ if

$$\forall t \in [0, T] : h(\varphi_t(x)) = h(\varphi_t(z)). \quad (6)$$

These indistinguishable pairs are collected in the (symmetric) set

$$\mathcal{I} = \{(x, z) \in \mathcal{M} \times \mathcal{M} \,|\, h(\varphi_t(x)) = h(\varphi_t(z))\}. \quad (7)$$

A system of the form (1) is *globally observable* if each state $x \in \mathcal{M}$ is only indistinguishable from itself, i. e., if $\mathcal{I}$ equals

$$\mathcal{J} = \big\{(x, z) \in \mathcal{M}^2 \,|\, x = z\big\}. \quad (8)$$

Such a system is called *locally observable at a point* $x_0 \in \mathcal{M}$ according to [2], [26] if there exists an open neighborhood $U_{x_0} \subset \mathcal{M}$ of x_0 such that

$$\mathcal{I} \cap U_{x_0}^2 = \big\{(x, z) \in U_{x_0}^2 \,|\, x = z\big\} = \mathcal{J} \cap U_{x_0}^2. \quad (9)$$

Note that this property is called *weakly observable* in [11], whose authors define local observability differently. Finally, the system is simply called *locally observable* if it is locally observable at any point in $\mathcal{M}$.

Until this point the indistinguishability is connected to equal output trajectories. Since this trajectory is locally analytic, the Lie series of such trajectories have equal coefficients. Therefore, we introduce the *observability map*

$$q(x) = \begin{pmatrix} h(x) \\ \mathrm{L}_f h(x) \\ \mathrm{L}_f^2 h(x) \\ \vdots \end{pmatrix} \quad (10)$$

and write equivalently

$$\mathcal{I} = \big\{(x, z) \in \mathcal{M}^2 \,|\, q(x) = q(z)\big\}. \quad (11)$$

As the Lie derivative of a polynomial along a polynomial vector field is a polynomial again, the set $\mathcal{I}$ is the zero set of polynomials. Thus, $\mathcal{I}$ is a real algebraic variety and global observability can be decided by means of algebraic geometry. As can be seen later, the space of not locally observable points is a variety, too.

D. Polynomial Ideals and their Varieties

A set of polynomials $I \subseteq \mathbb{R}[x]$ is called a *polynomial ideal* over the commutative ring $\mathbb{R}[x]$, if I fulfills the conditions

1) $0 \in I$
2) $a, b \in I \implies a + b \in I$
3) $a \in I, c \in \mathbb{R}[x] \implies ca \in I$.

Although those ideals contain, let alone the trivial ideal $\{0\}$, infinitely many polynomials, they are *generated* by a finite number of polynomials by Hilbert's basis theorem [6, p. 77]: One writes

$$\langle g_1, \ldots, g_s \rangle = \{a_1 g_1 + \cdots + a_s g_s \,|\, a_k \in \mathbb{R}[x]\} \quad (12)$$

for an ideal generated by $\{g_1, \ldots, g_s\}$.

The common zero set of all polynomials in an ideal I is called the *variety* of I. Herein, we are only interested in real zeros. Thus, we introduce the *real variety* of I to be the set of real points within the variety of I, denoted $\mathbf{var}^{\mathbb{R}}(I)$. For each ideal there is a unique (real) variety. However, the converse is not true. We assign to each real variety V an ideal $\mathbf{Ideal}(V)$ containing all polynomials that vanish on V. The relation

$$I \subseteq \mathbf{Ideal}\big(\mathbf{var}^{\mathbb{R}}(I)\big) \quad (13)$$

holds for any ideal I, although the equality holds not in general. The ideal $\mathbf{Ideal}\big(\mathbf{var}^{\mathbb{R}}(I)\big) = \mathbf{rad}^{\mathbb{R}}(I)$, referred as the *real radical*[1] of I, can be computed algebraically [5, p. 85] and equals

$$\Big\{g \,\big|\, g^{2m} + a \in I \text{ for some } m \in \mathbb{Z}_{>0},\ a \in \textstyle\sum \mathbb{R}[x]^2\Big\}, \quad (14)$$

where $\sum \mathbb{R}[x]^2$ denotes the cone of sums of squares of polynomials in $\mathbb{R}[x]$. If an ideal equals its real radical, the ideal is called *real*.

A few operations of ideals and varieties are required in the sequel: The *ideal sum*

$$I + J = \{g \,|\, g \in I \text{ or } g \in J\} \quad (15)$$

of I and J is an ideal that contains all polynomials contained in I or J. Geometrically, this corresponds to the intersection of varieties:

$$\mathbf{var}^{\mathbb{R}}(I + J) = \mathbf{var}^{\mathbb{R}}(I) \cap \mathbf{var}^{\mathbb{R}}(J). \quad (16)$$

The *intersection* $I \cap J$ contains all polynomials that are both contained in I and J:

$$I \cap J = \{g \,|\, g \in I \text{ and } g \in J\}, \quad (17)$$

corresponding to the union of varieties:

$$\mathbf{var}^{\mathbb{R}}(I \cap J) = \mathbf{var}^{\mathbb{R}}(I) \cup \mathbf{var}^{\mathbb{R}}(J). \quad (18)$$

Every real variety can be uniquely written as a union of *irreducible varieties*, i. e., varieties that cannot be written as a nontrivial union of smaller varieties. The corresponding real ideal can be written as an intersection of corresponding *prime*

[1] The notation $\sqrt[\mathbb{R}]{I}$ for the real radical of I is more common. The herein used notation is borrowed from [4].

ideals. The *saturation* $I : J^\infty$ *of* I *with respect to* J is the ideal

$$I : J^\infty = \{g \mid \exists m \in \mathbb{Z}_{\geq 0} : gh^m \in I \text{ for all } h \in J\}. \quad (19)$$

Geometrically the saturation is related to the difference set as follows:

$$\mathbf{var}^{\mathbb{R}}(I : J^\infty) = \overline{\mathbf{var}^{\mathbb{R}}(I) \setminus \mathbf{var}^{\mathbb{R}}(J)}, \quad (20)$$

where $\overline{V}$ denotes the Zariski closure [6, p. 190] of the set V, the smallest variety that contains V.

E. Lie Derivatives of Ideals

For an ideal $I \subseteq \mathbb{R}[x]$ we define the Lie derivative of I along a vector field f as the ideal

$$\mathrm{L}_f I = I + \langle \mathrm{L}_f g \mid g \in I \rangle. \quad (21)$$

Higher order Lie derivatives are again defined recursively by

$$\mathrm{L}_f^{k+1} I = \mathrm{L}_f \mathrm{L}_f^k I, \qquad \mathrm{L}_f^0 I = I. \quad (22)$$

These Lie derivatives form an ascending chain

$$I \subseteq \mathrm{L}_f I \subseteq \mathrm{L}_f^2 I \subseteq \cdots \subseteq \mathrm{L}_f^N I = \mathrm{L}_f^{N+1} I = \cdots = \mathrm{L}_f^\infty I, \quad (23)$$

which stabilizes as every ascending chain of ideals [6, p. 80]. The stabilized ideal in the chain of Lie derivatives will be called the *stabilized Lie derivative of I along f*. Note, that the stabilized Lie derivative can be defined without making use of the ascending chain at all [8], [9], as the saturation ideal can be defined without using a chain of ideal quotients [6, p. 202].

III. Observability Criteria

As noted before, the set $\mathcal{I}$ of indistinguishability state pairs is a variety of a polynomial ideal. The properties of this variety will be studied by means of the corresponding real ideal. This will lead to algebraic observability criteria.

First, the system (1) is extended by a copy of its own:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = F(x, z) := \begin{pmatrix} f(x) \\ f(z) \end{pmatrix} \quad (24a)$$

$$H(x, z) := h(x) - h(z) \quad (24b)$$

with the vector field $F : \mathcal{M}^2 \to \mathrm{T}\mathcal{M}^2$ and the (residuum) output map H. The observability map

$$Q(x, z) = \begin{pmatrix} \mathrm{L}_F^0 H(x, z) \\ \mathrm{L}_F^1 H(x, z) \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathrm{L}_f^0 h(x) - \mathrm{L}_f^0 h(z) \\ \mathrm{L}_f^1 h(x) - \mathrm{L}_f^1 h(z) \\ \vdots \end{pmatrix} \quad (25)$$

of the extended system happens to be exactly the difference $q(x) - q(z)$ of the original observability maps. With the ideal

$$\mathfrak{M} = \langle g_1(x), g_1(z), \ldots, g_m(x), g_m(z) \rangle \subseteq \mathbb{R}[x, z] \quad (26)$$

that corresponds to the variety $\{(x, z) \in \mathcal{M}^2\}$ and the ideal

$$\mathfrak{H} = \langle H_1, \ldots, H_p \rangle \subseteq \mathbb{R}[x, z] \quad (27)$$

generated by the vectorial components of the extended output map H, the variety (11) equals the real variety of the ideal

$$\mathfrak{I} = \mathbf{Ideal}(\mathcal{I}) = \mathbf{rad}^{\mathbb{R}}(\mathfrak{M} + \mathrm{L}_F^\infty \mathfrak{H}). \quad (28)$$

Note that the Lie derivative of all generators of $\mathfrak{M}$ are must be contained in $\mathfrak{M}$. Otherwise the vector field f in (1a) would not map into the tangent space $\mathrm{T}\mathcal{M}$ of $\mathcal{M}$. Thus, $\mathfrak{M}$ is already closed under the Lie derivative.

For the observability test, this ideal $\mathfrak{I}$ is compared with the ideal

$$\mathfrak{J} = \mathbf{Ideal}(\mathcal{J}) = \mathbf{rad}^{\mathbb{R}}(\langle x_1 - z_1, \ldots, x_n - z_n, g_1(x), \ldots, g_m(x) \rangle) \subseteq \mathbb{R}[x, z] \quad (29)$$

with $\mathcal{J}$ from (8).

A. Global Observability

Using the notation from before, we are able to state the following Theorem:

Theorem III.1. *The system* (1) *is globally observable if and only if* $\mathfrak{I} = \mathfrak{J}$.

This follows directly from the one-to-one correspondence between real ideals and real varieties and the definition of global observability.

B. Local Observability

The local observability is a bit more subtle. In addition to test local observability (at a particular point or for all points) we will compute the set of all locally observable points, or, more precisely, the set of points at that the system is not locally observable.

Theorem III.2. *With the notation from above the set of not locally observable points for* (1) *is the (real) variety*

$$N = \mathbf{var}^{\mathbb{R}}(((\mathfrak{I} : \mathfrak{J}) + \mathfrak{J}) \cap \mathbb{R}[x]), \quad (30)$$

where the elimination ideal is considered as an ideal in the ring $\mathbb{R}[x]$.

Proof. First, note that $N \subseteq \mathcal{M} = \mathbf{var}^{\mathbb{R}}(\mathfrak{J} \cap \mathbb{R}[x])$, such that only points in $\mathcal{M}$ need to be considered.

Assume system (1) is not locally observable at $x_0 \in \mathcal{M}$. Then there is $z \in U_{x_0}$ different from x_0, but arbitrary close, such that $(x_0, z) \in \mathcal{I}$. This implies that there exists an irreducible component $\mathcal{C}$ of the variety $\mathcal{I}$ that contains the point (x_0, z). This component must be different from $\mathcal{J}$ as $(x_0, z) \notin \mathcal{J}$. However, (x_0, z) lies arbitrary close to $(x_0, x_0) \in \mathcal{J}$. Thus, (x_0, x_0) is also contained in $\mathcal{C}$ as each variety is a closed set. This shows that the Zariski closure of $\mathcal{C} \setminus \mathcal{J}$ equals $\mathcal{C}$ again, and, as a consequence

$$(x_0, x_0) \in \overline{(\mathcal{I} \setminus \mathcal{J})} \cap \mathcal{J}. \quad (31)$$

The latter variety is precisely the variety of the ideal $(\mathfrak{I} : \mathfrak{J}) + \mathfrak{J}$, since $\mathfrak{I}$ is real. This shows that $x_0 \in N$.

If on the other hand (1) is locally observable at $x_0 \in \mathcal{M}$, all indistinguishable points in every neighborhood $U_{x_0}^2$

of $(x_0, x_0) \in \mathcal{M}^2$ are also contained in $\mathcal{J}$. Thus, the only irreducible component of $\mathcal{I}$ within $U^2_{x_0}$ is $\mathcal{J}$ itself. As a consequence,

$$(x_0, x_0) \notin \overline{\mathcal{I} \setminus \mathcal{J}}. \tag{32}$$

Since $\overline{(\mathcal{I} \setminus \mathcal{J})} \cap \mathcal{J}$ contains only points of the form (z, z), there is no point (x_0, z) for any z contained in this variety. So, $x_0 \notin N$. □

C. The Observable Subsystem

If the variety N of not locally observable points from Theorem III.2 is a subvariety of the state-space $\mathcal{M}$, the dimension of N is neccesary smaller the that of $\mathcal{M}$. Thus, nearly all points in the state-space are locally observable. If one writes the variety $\mathcal{I}$ of indistinguishable pairs as a union

$$\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_s \tag{33}$$

of irreducible varieties, only one of the irreducible components can contain $\mathcal{J}$, the variety of equal state pairs. This component must be furthermore equal to $\mathcal{J}$. If in contrast one such component would strictly contain $\mathcal{J}$, it would do so in any neighborhood $U^2_{x_0}$ and (9) would fail. Since all components $\mathcal{I}_i$, $i = 1, \ldots, s$ that contain $\mathcal{J}$ are equal to $\mathcal{J}$, such a variety can occur only once in the irreducible decomposition.

This holds also for the locally (at every point) and globally observable case.

From Theorem III.2 one obtains the set of points, where the observability map is not locally invertible.

If in contrast $N = \mathcal{M}$, i. e., the system is a no point locally observable, the statement from Theorem III.2 is to some extend unsatisfying. In this case there must be irreducible varieties in the decomposition (33) that strictly contain $\mathcal{J}$. Assume for a moment that exactly one such component exists, say $\mathcal{I}_1$. For any fixed $x_0 \in \mathcal{M}$ only this variety contains all indistinguishable states (x, x_0) in any neighborhood of (x_0, x_0). This means that there is a positive dimensional not observable part.

Apart from this ambiguity there may be points in the state-space where even the observable subsystem is not locally observable, namely where the other components $\mathcal{I}_2, \ldots, \mathcal{I}_s$ intersect with $\mathcal{J}$. These points can be computed the same way and with the same arguments as in Theorem III.2, with $\mathfrak{I}$ replaced by the corresponding ideal $\mathfrak{I}_2 \cap \cdots \cap \mathfrak{I}_s$ of the corresponding minimal decomposition. By the previously made assumption this variety is now strictly contained in $\mathcal{M}$ and of smaller dimension.

Except for that points, the system can be locally decomposed into an observable subsystem and a not observable part. This transformation is sketched in the sequel using the ideal corresponding to the indistinguishable states, at least for the observable part.

The observable subsystem can be obtained by integrating the (involutive) codistribution $\mathrm{Span}\{\mathrm{d}q\}$ spanned by the obserability map [18, pp. 95], where

$$\mathrm{d}q = \tfrac{\partial q}{\partial x_1}\mathrm{d}x_1 + \cdots + \tfrac{\partial q}{\partial x_n}\mathrm{d}x_n. \tag{34}$$

Since all components (or their multiples) of the extended observability map are contained in $\mathfrak{I}$, we have $\mathrm{Span}\{\mathrm{d}q\} \subseteq \mathrm{d}\mathfrak{I} = \mathrm{Span}\{\mathrm{d}p \,|\, p \in \mathfrak{I}\}$. Note that the latter distribution may be greater since it contains the derivatives of the defining polynomials g_i of the manifold $\mathcal{M}$. Furthermore, using the generators of the ideal may avoid zeros that do not originate from observability properties, see Subsection IV-A.

Using the ideal

$$\mathfrak{I} = \langle p_1(x, z), \ldots, p_k(x, z)\rangle \tag{35}$$

one can substitute new variables

$$\xi_1 = p_1(x, x_0), \ \ldots, \ \xi_k = p_k(x, x_0) \tag{36}$$

with any fixed x_0. These new variables may not be independent, not alone because of the constraints arising from the embedding. Constraints for the new variables can be found by computing the elimination ideal

$$\Xi = \mathbb{R}[\xi] \cap \langle \xi_1 - p_1(x, x_0), \ldots, \xi_k - p_k(x, x_0), g_1(x), \ldots, g_m(x)\rangle \tag{37}$$

by eliminating the variables x. All polynomials in this ideal must evaluate to zero for all ξ. An easy method to obtain the transformed output map (1b) is to include the polynomial $h(x) - h(z) \in \mathfrak{I}$ in the generating set of $\mathfrak{I}$. Then one gets $y = \xi_i$ for some $i \in \{1, \ldots, k\}$. Finally, the differential equations for the new coordinates read

$$\dot{\xi}_i = \mathrm{L}_f p_i(x, x_0) \tag{38}$$

with the value of x taken from the inverse transform of (36), which should always be possible. However, it is not known if this leads to the form (1a) with polynomial components of the transformed vector field in the variables ξ. Although starting from some particular order all higher-order Lie derivatives of the output map $h(x)$ can be written as $\mathbb{R}[x]$-linear combinations of the lower-order ones, it is not guaranteed that these can be written by sums and products of the lower-order Lie derivatives itself.

It remains to show that the previously made assumption that only one irreducible component of $\mathcal{I}$ can contain $\mathcal{J}$. If not, at least two different varieties would intersect and the intersection contains $\mathcal{J}$. Thus, all polynomials in $\mathfrak{I}$ must have a common zero with their derivative at equal points (x_0, x_0). This implies that $\mathrm{d}q$ is identically zero and that the observable subsystem is empty. In this case the two intersecting varieties must be $\mathcal{M} \times \mathcal{M}$ and are equal, a contradiction to the decomposition being irreducible.

This fact allows to carry out the computation of the observable subsystem with the prime ideal $\mathfrak{I}_1$, which is the only one contained in $\mathfrak{J}$, instead of $\mathfrak{I}$.

IV. EXAMPLES

A. Academic Example 1

As a first example consider the system

$$\begin{aligned} \dot{x} &= 0, \quad x \in \mathbb{R} \\ y &= x^3 \end{aligned} \tag{39}$$

from [18, p. 99]. This system is clearly globally observable, since the output map $h(x) = x^3$ is injective. This will be shown using the method described herein:

Here, any Lie derivative of the output map is identically zero. Thus, with the notation used herein, one has $\mathfrak{H} = \langle x^3 - z^3 \rangle = \mathrm{L}_F^\infty \mathfrak{H}$. This ideal is not real, which can be shown as follows: We have $(x-z)(x^3-z^3) \in \mathrm{L}_F^\infty \mathfrak{H}$. This polynomial can be written as

$$(x-z)^2(x^2+z^2-xz) = (x-z)^2\Big(\big(\tfrac{1}{2}x - z\big)^2 + \tfrac{3}{4}x^2\Big) = \big((x-z)\big(\tfrac{1}{2}x - z\big)\big)^2 + \Big(\tfrac{\sqrt{3}}{2}(x-z)x\Big)^2, \quad (40)$$

which is a sum of squares. Thus, $x(x-z)$ is contained in $\mathbf{rad}^{\mathbb{R}}(\mathrm{L}_F^\infty \mathfrak{H})$. By the same argument, $z(x-z)$ is also contained, and so is their difference $(x-z)^2$. Because the latter polynomial is a square, too, we arrive at $x - z \in \mathbf{rad}^{\mathbb{R}}(\mathrm{L}_F^\infty \mathfrak{H})$. Thus, we have

$$\mathfrak{I} = \mathbf{rad}^{\mathbb{R}}(\mathrm{L}_F^\infty \mathfrak{H}) = \langle x - z \rangle = \mathfrak{J} \quad (41)$$

and the global observability follows.

This cannot be shown directly using the observation space $O = \mathrm{Span}\{x^3\}$, as $\mathrm{d}O(x) = \mathrm{Span}\{3x^2\}$ is singular at $x = 0$.

B. Academic Example 2

Consider the non-observable system

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \\ y &= h(x) = x_1^2 + x_2^2 \end{aligned} \quad (42)$$

with $\mathcal{M} = \mathbb{R}^2$. Here, $\mathfrak{H} = \mathrm{L}_F^\infty \mathfrak{H} = \langle x_1^2 + x_2^2 - z_1^2 - z_2^2 \rangle$, since the Lie derivative of the output map is a multiple of the latter again. This ideal is already radical such that

$$\mathfrak{I} = \langle x_1^2 + x_2^2 - z_1^2 - z_2^2 \rangle \subsetneq \mathfrak{J} = \langle x_1 - z_1, x_2 - z_2 \rangle. \quad (43)$$

Thus, the system is not globally observable.

In order to test local observability we compute the ideal quotient $\mathfrak{I} : \mathfrak{J} = \mathfrak{I}$. Consequently, the set $N = \mathbb{R}^2$ of not locally observable points equals the whole state-space. This indicates that a decomposition into an observable subsystem and a not observable part is possible:

The ideal $\mathfrak{I}$ is a principal prime ideal, i. e., it is generated by a single polynomial and it cannot be written as a nontrivial intersection. Thus, the observable part can be described by a single new variable $\xi_1 = x_1^2 + x_2^2$, which obeys the differential equation

$$\dot{\xi}_1 = \mathrm{L}_f(x_1^2 + x_2^2) = 2(x_1^2 + x_2^2) = 2\xi_1. \quad (44)$$

As $\mathrm{d}\xi_1 = 2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2$, there must be another variable ξ_2 with $\mathrm{d}\xi_2 = -x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2$ in the annihilator of $\mathrm{d}\xi_1$. This differential equation is solved by $\xi_2 = \arctan\big(\frac{x_2}{x_1}\big)$, which can be algebraically continued for all states except $(x_1, x_2) = (0, 0)$. Although this tranformation is not a polynomial one, the corresponding differential equation takes the simple form $\dot{\xi}_2 = 0$.

C. Academic Example 3

We consider the system

$$\begin{aligned} \dot{x} &= f(x) = \begin{pmatrix} -x_2 + x_1(1 - x_1^2 - x_2^2) \\ x_1 + x_2(1 - x_1^2 - x_2^2) \\ -x_3(x_1^2 + x_2^2) \end{pmatrix} \\ y &= h(x) = x_1^2 + x_2^2 + x_3^2. \end{aligned} \quad (45)$$

from [7], [20], [23] defined on the manifold $\mathcal{M} = \mathbb{R}^3 \ni x$. The stabilized Lie derivative of the ideal

$$\mathfrak{H} = \langle H \rangle = \langle x_1^2 + x_2^2 + x_3^2 - z_1^2 - z_2^2 - z_3^2 \rangle \quad (46)$$

is easily computed:

$$\begin{aligned} &\langle x_3 - z_3, x_1^2 + x_2^2 - z_1^2 - z_2^2 \rangle \cap \\ &\quad \langle x_3 + z_3, x_1^2 + x_2^2 - z_1^2 - z_2^2 \rangle \cap \\ &\quad\quad \langle z_1^2 + z_2^2 + z_3^2 - 1, x_1^2 + x_2^2 + x_3^2 - 1 \rangle \end{aligned} \quad (47)$$

as the chain (23) already stabilizes after the second ideal. Note that there is no algebraic constraint on the coordinates such that $\mathfrak{M} = \langle 0 \rangle$, with the notation from above, and thus, $\mathrm{L}_F^\infty \mathfrak{H} + \mathfrak{M} = \mathrm{L}_F^\infty \mathfrak{H}$. The real radical of the latter ideal is found to equal

$$\begin{aligned} \mathfrak{I} = \mathbf{rad}^{\mathbb{R}}(\mathrm{L}_F^\infty \mathfrak{H}) = &\langle x_3 - z_3, x_1^2 + x_2^2 - z_1^2 - z_2^2 \rangle \cap \\ &\langle x_3 + z_3, x_1^2 + x_2^2 - z_1^2 - z_2^2 \rangle \end{aligned} \quad (48)$$

Thus, by Theorem III.1, the system is not globally observable.

We compute the set of not locally observable points using Theorem III.2 as follows: The ideal quotient $\mathfrak{I} : \mathfrak{J}$ yields the same ideal $\mathfrak{I}$ again. Thus, the hyperplane $x = z$ is contained in the variety $\mathbf{var}^{\mathbb{R}}(\mathfrak{I})$, but also its neighborhood. Consequently, the variety N of not locally observable points turns out to be $\mathbb{R}^3$, the whole state-space.

Since the system is not observable in the whole state-space, there exists a decomposition into an observable and a not observable subsystem. In the decomposition (48) only the first ideal in the intersection is contained in $\mathfrak{J}$. This means that only the variety corresponding to this ideal contains all equal pairs $x = z$ of states. From the two generators new variables

$$\begin{aligned} \xi_1 &= x_3 \\ \xi_2 &= x_1^2 + x_2^2 \end{aligned} \quad (49)$$

are introduced. These obey the differential equations

$$\begin{aligned} \dot{\xi}_1 &= \mathrm{L}_f x_3 = -x_3(x_1^2 + x_2^2) = -\xi_1\xi_2 \\ \dot{\xi}_2 &= \mathrm{L}_f(x_1^2 + x_2^2) = (x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) = \xi_2(1 - \xi_2) \end{aligned} \quad (50)$$

and are algebraically independent, as

$$\langle \xi_1 - x_3, \xi_2 - x_1^2 - x_2^2 \rangle \cap \mathbb{R}[\xi_1, \xi_2] = \langle 0 \rangle. \quad (51)$$

The corresponding output map simply becomes $y = \xi_1^2 + \xi_2$.

As in the example before, the annihilator of $\mathrm{d}\xi_1 = \mathrm{d}x_3$ and $\mathrm{d}\xi_2 = 2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2$ equals $-x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2$. Thus, the remaining variable $\xi_3 = \arctan\big(\frac{x_2}{x_1}\big)$ is introduced, which satisfies

$$\dot{\xi}_3 = \mathrm{L}_f \arctan\big(\tfrac{x_2}{x_1}\big) = 1. \quad (52)$$

In this form the observability character becomes evident.

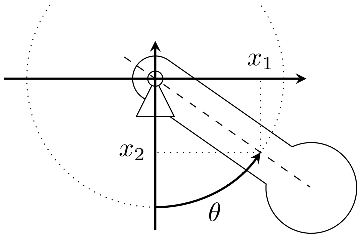

Figure 1. The pendulum with the coordinates used herein.

D. The Pendulum

Consider the movement of a rigid pendulum in a plane as depicted in Figure 1. As coordinates we use the (redundant) Cartesian coordinates $(x_1, x_2) = (\sin\theta, -\cos\theta)$ on the unit circle for the pendulums deflection, and the angular velocity $x_3 = \dot{\theta}$, on the manifold $\mathcal{M} = S^1 \times \mathbb{R}$. Using furthermore dimensionless quantities, the equations of motion read

$$\dot{x} = f(x) = \begin{pmatrix} -x_2 x_3 \\ x_1 x_3 \\ -x_1 \end{pmatrix} \tag{53a}$$

$$0 = g(x) = x_1^2 + x_2^2 - 1. \tag{53b}$$

For this system several output maps are considered.

First, consider the measurement of the angular velocity $x_3 = h(x)$. Using in addition a copy of that system we compute the ideal $\mathrm{L}_F^\infty \mathfrak{H}$ starting with

$$\mathfrak{H} = \langle x_3 - z_3 \rangle. \tag{54}$$

Adding the Lie derivatives of the generators yields the chain of ideals

$$\begin{aligned} \mathfrak{H} \subset \langle x_1 - z_1, x_3 - z_3 \rangle \subset \\ \langle x_1 - z_1, x_3 - z_3, x_2x_3 - z_2z_3 \rangle \subset \\ \big\langle x_1 - z_1, x_3 - z_3, x_2x_3 - z_2z_3, \\ x_1x_3^2 - x_1x_2 - z_1z_3^2 + z_1z_2 \big\rangle = \mathrm{L}_F^\infty \mathfrak{H}. \end{aligned} \tag{55}$$

In combination with

$$\mathfrak{M} = \big\langle x_1^2 + x_2^2 - 1, z_1^2 + z_2^2 - 1 \big\rangle \tag{56}$$

one obtains the primary decomposition of the ideal

$$\begin{aligned} \mathfrak{M} + \mathrm{L}_F^\infty \mathfrak{H} = \langle x_1, x_2 - 1, x_3, z_1, z_2 + 1, z_3 \rangle \cap \\ \langle x_1, x_2 + 1, x_3, z_1, z_2 - 1, z_3 \rangle \cap \\ \big\langle x_1 - z_1, x_2 - z_2, x_3 - z_3, x_1^2 + x_2^2 - 1 \big\rangle, \end{aligned} \tag{57}$$

which is real and thus equals $\mathfrak{I}$. The last ideal occurring in the primary decomposition is precisely $\mathfrak{J}$, the ideal corresponding to the variety (8) of equal points for each system copy. The remaining ideals correspond to irreducible varieties that contain a single point, each. In such a point the two pendulums are in different equilibrium positions, which cannot be distinguished, namely one pointing up and the other one pointing down. Nonetheless, as these two equilibria share no points with the variety $\mathcal{J}$ of identical states, one computes $N = \varnothing \subseteq \mathcal{M}$ for the set of not locally observable points by Theorem III.2. However, as $\mathfrak{I} \neq \mathfrak{J}$, the system is not globally observable.

We discuss another output map $h(x) = x_3^2 - x_2$ corresdonding to the tension force of the pendulum. In this case Lie derivatives of the output map up to order four are required in order to stabilize the chain. We omit the details and state the stabilized ideal

$$\begin{aligned} \mathfrak{M} + \mathrm{L}_F^\infty \mathfrak{H} = \big\langle x_1^2 + x_2^2 - 1, x_1 - z_1, x_2 - z_2, x_3 - z_3 \big\rangle \\ \cap \big\langle x_1^2 + x_2^2 - 1, x_1 + z_1, x_2 - z_2, x_3 + z_3 \big\rangle, \end{aligned} \tag{58}$$

which equals $\mathfrak{I}$ since it is also real. Again, the system is not globally observable with this output map. The first ideal occuring in the intersection above is again $\mathfrak{J}$, the ideal of all polynomials that vanish on equal state pairs. As the second one is not contained in $\mathfrak{J}$ one has

$$\mathfrak{I} : \mathfrak{J} = \big\langle x_1^2 + x_2^2 - 1, x_1 + z_1, x_2 - z_2, x_3 + z_3 \big\rangle \tag{59}$$

and thus

$$(\mathfrak{I} : \mathfrak{J}) + \mathfrak{J} = \big\langle x_2^2 - 1, x_1, z_1, x_2 - z_2, x_3, z_3 \big\rangle. \tag{60}$$

Eliminating the auxillary variables z of the system copy leaves one with the ideal

$$\big\langle x_1, x_2^2 - 1, x_3 \big\rangle. \tag{61}$$

The variety

$$N = \{(0, -1, 0), (0, 1, 0)\} \tag{62}$$

of the latter ideal is the set of the equilibrium points, corresponding to $\theta = k\pi$ with $k \in \mathbb{Z}$.

E. The Thomas' cyclically symmetric attractor

The Thomas attractor [27] is a third order system

$$\begin{aligned} \dot{X}_1 &= \sin(X_2) - bX_1 \\ \dot{X}_2 &= \sin(X_3) - bX_2 \\ \dot{X}_3 &= \sin(X_1) - bX_3 \end{aligned} \tag{63}$$

that posesses chaotic behaviour, depending on the value of the parameter b. A particulary interesting case is $b = 0$, for which observability will be studied herein. Using the same technique as before, this system can be embedded into the six-dimensional euclidean space using the transformation

$$\begin{aligned} x_1 &= \cos X_1, \quad x_2 = \sin X_1 \\ x_3 &= \cos X_2, \quad x_4 = \sin X_2 \\ x_5 &= \cos X_3, \quad x_6 = \sin X_3. \end{aligned} \tag{64}$$

Consequently, the newly introduced redundant coordinates must fulfill the constraints

$$\begin{aligned} g_1(x) &= x_1^2 + x_2^2 - 1 = 0 \\ g_2(x) &= x_3^2 + x_4^2 - 1 = 0 \\ g_3(x) &= x_5^2 + x_6^2 - 1 = 0 \end{aligned} \tag{65}$$

and one has the ideal

$$\mathfrak{M} = \langle g_1(x), g_2(x), g_3(x), g_1(z), g_2(z), g_3(z)\rangle. \tag{66}$$

The newly introduced coordinates follow the differential equations

$$\dot{x} = f(x) = \begin{pmatrix} -x_2x_4 \\ x_1x_4 \\ -x_4x_6 \\ x_3x_6 \\ -x_6x_2 \\ x_5x_2 \end{pmatrix}. \tag{67}$$

If we consider the output maps $h_1(x) = x_1, h_2(x) = x_2$, i. e., measurement of a particular original state variable X_1 (up to multiples of 2π), the stabilized Lie derivative of $\mathfrak{H} = \langle x_1 - z_1, x_2 - z_2\rangle$ is to be computed. We omit intermediate steps and directly give the resulting ideal $\mathfrak{I}$, which can be written as an intersection of 18 irreducible ideals, one of which is $\mathfrak{J}$. Another ideal of these is

$$\mathfrak{M} + \langle x_1 - z_1, x_2 - z_2, x_3 + z_3, x_4 - z_4, x_5 + z_5, x_6 + z_6\rangle. \tag{68}$$

The remaining 16 ideals can be written as

$$\langle x_1 - z_1, x_2, z_2, x_3 \pm 1, z_3 \pm 1, x_4, z_4, x_5 \pm 1, z_5 \pm 5, x_6, z_6\rangle \tag{69}$$

with arbitrary combinations of the signs.

As such, the system is not globally observable. However, since

$$(\mathfrak{I} : \mathfrak{J}) + \mathfrak{J} = \langle 1\rangle = \mathbb{R}[x, z], \tag{70}$$

the set of not locally observable points is empty such that one has shown local observability (on the whole manifold).

V. Conclusion

A criterion to test global and local observability of a polynomial dynamical system has been given. In addition, all points in the state space at that the system fails to be locally observable can be computed. Currently, this criterion can be applied only to polynomial systems of the form (1). While some nonlinear systems can be formulated in this form, some fall not into this class. Therefore, it is desirable to extend the system class. Some open questions remain regarding the decomposition with respect to observability. In particular if the observable part can be written in the same form as the whole system and if there exist a method to handle the not observable part algebraically.

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