Sparse Filtering Under Norm-Bounded Exogenous Disturbances Using Observers

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***Abstract*—The paper considers the sparse filtering problem under arbitrary norm-bounded exogenous disturbances. We propose a simple and universal observer-based approach to its solution, based on the LMI technique and the method of invariant ellipsoids; it allows the use of a reduced number of system outputs. From a technical point of view of application, we reduce the original problem to semi-definite programming, which is easily solved numerically. The proposed simple approach is easy to implement and can be equally extended to systems in continuous and discrete time.**

***Index Terms*—linear system, filtering, sparsity, exogenous disturbances, linear matrix inequalities, invariant ellipsoids**

I. Introduction

In the modern literature, the term *sparse filtering* is mainly assigned to such areas as machine learning, pattern recognition, signal and image processing; see, for example, [1]–[5]. In many situations, the classical assumption that the disturbances are random is not justified. Frequently, it is known that the disturbances are bounded only. In this case, *guaranteed* estimates of states can be constructed. This approach was proposed in the works of Witzenhausen, Bertsekas and Rhodes, Schweppe [6]. At about the same time, similar problems were developed by such researchers as Kurzhansky [7]. A significant contribution to this circle of research was made by Chernousko [8].

In the papers [9], [10], the problem of filtering with nonrandom bounded exogenous disturbances was considered, but only for stationary problem statements. Moreover, a state estimate was sought such that its residual is guaranteed to be enclosed in a single so-called *invariant ellipsoid*. The filter was also sought as the linear stationary filter. In this class, the problem turned out to be completely solvable, so that it was possible to construct an optimal filter and state estimate. From a technical point of view, the LMI apparatus [11] was used in [9], [10]. The LMI technique has proven itself well in the analysis and design (see, e.g. [12], [13]), but has not been widely used in filtering problems. A systematic presentation of this technique is given in the monograph [14].

On the other hand, the *sparsity* ideas are widely used in the various fields (e.g., see [15], [16]), but not in control. We mention publications [17], [18] devoted to the sparse feedback design. In [19], a new approach to constructing a sparse feedback was proposed, which is associated with minimizing nonzero rows or nonzero columns of the matrix. Such matrices are called row-sparse and column-sparse, respectively.

This method is distinguished by simplicity: the initial problems are reduced to solving low-dimensional convex optimization problem, and for its numerical solution one can use standard tools, such as MATLAB-based package YALMIP [20] and cvx [21], [22]. We mention the versatility of the proposed approach as continuous- and discrete-time cases are considered uniformly, and it is applicable to both linear state and output feedback design. At last, we stress its extendability to the various robust formulations, as well as to the optimal control problems, etc.

This paper is a natural continuation of [9], [10], and [19]. It proposes an approach to the solution of the *sparse filtering problem*, that is, filtering using a reduced number of outputs in the presence of arbitrary bounded exogenous disturbances.

Throughout the following, $\|\cdot\|$ is the Euclidean norm of a vector and the spectral norm of a matrix, $^{\mathrm{T}}$ is the transposition symbol, $\mathbb{S}^{n\times n}$ is the class of symmetric real $n\times n$ matrices, I is the identity matrix of appropriate dimension, and all matrix inequalities are understood in the sense of the sign definiteness of the matrices.

The present paper is the revised and expanded version of talk [23] presented at the IEEE 25th International Conference on System Theory, Control and Computing (ICSTCC 2021). In particular, a number of additions have been made to the text of the article, and the list of references has been significantly expanded and updated.

II. Sparse Control

Let us recall the main ideas of the above mentioned approach to the construction of sparse control. Let $\Omega \in \mathbb{R}^{n\times p}$;

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we introduce into consideration the following matrix norms:

$$\|\Omega\|_{r_1} = \sum_{i=1}^{n} \max_{1\le j\le p} |\omega_{ij}|, \quad \|\Omega\|_{c_1} = \sum_{j=1}^{p} \max_{1\le i\le n} |\omega_{ij}|.$$

The following result stated in [19].

Theorem 1: If the problem

$$\min \|\Omega\|_{r_1} \quad \text{s.t.} \quad A\Omega = B,$$

where $A \in \mathbb{R}^{m\times n}$, $m < n$, $B \in \mathbb{R}^{m\times p}$, $\Omega \in \mathbb{R}^{n\times p}$, is feasible, then there exists a solution with no more than m nonzero rows.

A similar result can be stated for the c_1-norm.

The approach developed in [19] allows the regular design of sparse controls in various statements. In particular, consider the linear system in continuous time

$$\dot{x} = Ax + Bu \tag{1}$$

with state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$, i.e. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$; the pair (A, B) is controllable.

The goal is to construct a *sparse* stabilizing control

$$u = \Phi x,$$

in the sense of zero components of the control vector. So, we are interesting in finding the *row-sparse* stabilizing controller $\Phi \in \mathbb{R}^{m\times n}$, i.e. having zero rows.

The technique required to obtain this result will be used in the sequel. It is well known, the matrix $A + B\Phi$ is stable iff there exists matrix $\Omega \succ 0$ such that

$$(A + B\Phi)^{\mathrm{T}}\Omega + \Omega(A + B\Phi) \prec 0.$$

Pre-multiplying and post-multiplying this inequality by the matrix $\Xi = \Omega^{-1}$ we obtain the inequality

$$A\Xi + \Xi A^{\mathrm{T}} + B\Phi\Xi + \Xi\Phi^{\mathrm{T}}B^{\mathrm{T}} \prec 0.$$

Finally, introducing a new matrix variable $\Psi = \Phi\Xi$, we obtain the LMI

$$A\Xi + \Xi A^{\mathrm{T}} + B\Psi + \Psi^{\mathrm{T}}B^{\mathrm{T}} \prec 0, \quad \Xi \succ 0, \tag{2}$$

in the matrix variables $\Xi \in \mathbb{S}^{n\times n}$ and $\Psi \in \mathbb{R}^{m\times n}$. Therefore, any stabilizing gain matrix for system (1) is presented by the expression

$$\widehat{\Phi} = \widehat{\Psi}\widehat{\Xi}^{-1}$$

where the matrices $\widehat{\Xi}$ and $\widehat{\Psi}$ satisfy (2).

It is clear, right multiplication preserves the row-sparse structure of the matrix. Therefore, if the solution $\widehat{\Psi}$ of the linear matrix inequality (2) is row-sparse, then the gain matrix $\widehat{\Phi}$ is row-sparse. Hence, the row sparsity of the matrix Ψ can be achieved by minimizing its r_1-norm. Thus, the following statement holds.

Statement 1 ([19]): The solution $\widehat{\Xi}$ and $\widehat{\Psi}$ of the convex optimization problem

$$\min \|\Psi\|_{r_1} \quad \text{s.t.} \quad A\Xi + \Xi A^{\mathrm{T}} + B\Psi + \Psi^{\mathrm{T}}B^{\mathrm{T}} \prec 0, \quad \Xi \succ 0,$$

in the matrix variables $\Xi \in \mathbb{S}^{n\times n}$ and $\Psi \in \mathbb{R}^{m\times n}$, defines the row-sparse stabilizing gain matrix

$$\Phi_{\text{sp}} = \widehat{\Psi}\widehat{\Xi}^{-1}$$

for system (1).

With Statement 1, we detect the stabilizing control inputs. These controls are determined by nonzero rows of the matrix Φ_{sp}. Evidently, we can not state that the resulting solution will be row-sparse, but it is expected by virtue of Theorem 1.

The author apply these ideas to the sparse filtering problem stated in the next section.

III. Continuous-Time Case

A. *Filtering problem*

Consider the dynamical system

$$\begin{aligned} \dot{x} &= Ax + B\nu, \quad x(0) = x_0, \\ y &= Cx + D\nu, \end{aligned} \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{l\times n}$, with state $x(t) \in \mathbb{R}^n$, observed output $y(t) \in \mathbb{R}^l$, and exogenous disturbances $\nu(t) \in \mathbb{R}^m$ satisfying the constraint

$$\|\nu(t)\| \le 1 \quad \text{for all } t \ge 0; \tag{4}$$

the pair (A, B) is controllable and the pair (A, C) is observable. Let the state x of system (3) be unavailable, and the information about the system is provided by its output y.

We construct a linear filter described by the differential equation

$$\dot{\widehat{x}} = A\widehat{x} + \mathcal{F}(y - C\widehat{x}), \quad \widehat{x}(0) = 0.$$

We emphasize that only the constant matrix $\mathcal{F} \in \mathbb{R}^{n\times l}$ is to be chosen.

The goal is to find the minimal (in the certain sense) invariant ellipsoid containing the residual

$$\rho(t) = x(t) - \widehat{x}(t).$$

The application of the ideology of invariant ellipsoids to control systems is described in [11], [14] in detail. Recall that the ellipsoid

$$\mathcal{V}_x = \left\{x \in \mathbb{R}^n : \quad x^{\mathrm{T}}\Xi^{-1}x \le 1\right\}, \quad \Xi \succ 0,$$

is called invariant for a dynamical system if the condition $x(0) \in \mathcal{V}_x$ yields $x(t) \in \mathcal{V}_x$ for all times $t \ge 0$. So, any trajectory of the system, starting from any point lying inside the ellipsoid $\mathcal{V}_x$, at any time instant will be in this ellipsoid for all admissible exogenous disturbances.

By virtue of the attractiveness property of an invariant ellipsoid, the filtering accuracy is asymptotic for large deviations, and the filtering accuracy is uniform in t for small deviations.

There are many invariant ellipsoids, the goal is to find the minimum one and, to minimize it over $\mathcal{F}$. It is convenient for us to assume that the *minimal* ellipsoid has the minimal trace of its matrix. In [9], the following result was stated.

Theorem 2: Let $\widehat{\Omega}$ and $\widehat{\Psi}$ be the solution of the optimization problem

$$\min \operatorname{tr} \Upsilon$$

under the constraints

$$\begin{pmatrix} A^{\mathrm{T}}\Omega + \Omega A - \Psi C - C^{\mathrm{T}}\Psi^{\mathrm{T}} + \mu\Omega & \Omega B - \Psi D \\ B_1^{\mathrm{T}}\Omega - B_2^{\mathrm{T}}\Psi^{\mathrm{T}} & -\mu I \end{pmatrix} \preceq 0,$$

$$\begin{pmatrix} \Upsilon & I \\ I & \Omega \end{pmatrix} \succeq 0, \quad \Omega \succ 0,$$

with the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $\mu > 0$.

Then the optimal filter matrix gives as

$$\widehat{\mathcal{F}} = \widehat{\Omega}^{-1}\widehat{\Psi},$$

and minimal invariant ellipsoid for the residual of (3) with $x_0 = 0$ defined by the matrix

$$\widehat{\Xi} = \widehat{\Omega}^{-1}.$$

B. Sparse filtering

We will seek a sparse solution of the filtering problem for system (3), (4). Note, that the filter matrix $\mathcal{F}$ has the form

$$\mathcal{F} = \Omega^{-1}\Psi.$$

Therefore, if the matrix Ψ be column-sparse, then the corresponding filter matrix $\mathcal{F}$ be column-sparse as well. In turn, the column sparsity of the matrix Ψ can be achieved by minimizing its c_1-norm.

Thus, we have the following algorithm, which involves the execution of three consecutive steps.

Algorithm 1:

Step 1. Solving the optimization problem

$$\min \operatorname{tr} \Upsilon \tag{5}$$

under the constraints

$$\begin{pmatrix} A^{\mathrm{T}}\Omega + \Omega A - \Psi C - C^{\mathrm{T}}\Psi^{\mathrm{T}} + \mu\Omega & \Omega B - \Psi D \\ B_1^{\mathrm{T}}\Omega - B_2^{\mathrm{T}}\Psi^{\mathrm{T}} & -\mu I \end{pmatrix} \preceq 0, \tag{6}$$

$$\begin{pmatrix} \Upsilon & I \\ I & \Omega \end{pmatrix} \succeq 0, \quad \Omega \succ 0, \tag{7}$$

in the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $\mu > 0$, we obtain the values Ω^*, Ψ^*, and Υ^* which define the matrix

$$\mathcal{F}^* = (\Omega^*)^{-1}\Psi^*$$

of the optimal filter, and the matrix

$$\Xi^* = (\Omega^*)^{-1}$$

of the minimal invariant ellipsoid for the residual, and the corresponding value

$$\mathcal{J}^* = \operatorname{tr} \Upsilon^*$$

of the cost function.

Step 2. Having the value $\mathcal{J}^*$, we implement the relaxation coefficient $\lambda > 1$ and consider c_1-optimization problem

$$\min \|\Psi\|_{c_1} \quad \text{s.t. (6), (7) and } \operatorname{tr} \Upsilon \preceq \lambda\mathcal{J}^* \tag{8}$$

in the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $\mu > 0$.

By virtue of the properties of the c_1-norm, one can expect the occurrence of zero columns in the solution $\widehat{\Psi}_0$ of this problem.

Step 3. We resolve the problem (5)–(7) with the additional constraint that the matrix variable Ψ has zero columns at the same places as the matrix $\widehat{\Psi}_0$. Its solution $\widehat{\Omega}$, $\widehat{\Psi}$ defines the column-sparse filter matrix

$$\widehat{\mathcal{F}} = \widehat{\Omega}^{-1}\widehat{\Psi}$$

and the matrix $\widehat{\Xi} = \widehat{\Omega}^{-1}$ of the corresponding invariant ellipsoid for the residual.

In section V, it will be shown by example that the proposed procedure leads to highly sparse matrices of the filter with small losses in terms of the cost criterion.

Remark 1: If we have a priori information about the initial condition $x(0) \in \mathcal{V}_0$ of the system, where

$$\mathcal{V}_0 = \{x \in \mathbb{R}^n: \quad x^{\mathrm{T}}\Xi_0^{-1}x \leq 1\}.$$

Then, letting $\widehat{x}(0) = 0$, we can guarantee that $\rho(0) \in \mathcal{V}_0$. If we prescribe that

$$\mathcal{V}_0 \subset \mathcal{V},$$

then we can guarantee that $\rho(t) \in \mathcal{V}$ for all $t \geq 0$.

Accordingly, if we add the condition

$$\Omega \preceq \Xi_0^{-1}$$

to the constraints (6)–(7) in Algorithm 1, then we obtain not only asymptotic, but uniform estimate of the sparse filtering accuracy.

Remark 2: Often it is necessary to evaluate the quality of filtering not all coordinates of the state x, but only some of coordinates. Let us have the output

$$z = C_z x$$

and the goal is to make the residual of its estimate

$$\rho_z = z - \widehat{z} = C_z(x - \widehat{x})$$

as small as possible. The solution of this problem is achieved by replacing the first of the conditions (7) by

$$\begin{pmatrix} \Upsilon & C_z \\ C_z^{\mathrm{T}} & \Omega \end{pmatrix} \succeq 0.$$

IV. Discrete-Time Case

The analogous results can be established for the dynamical system

$$\begin{aligned} x_{k+1} &= Ax_k + B\nu_k, \\ y_k &= Cx_k + D\nu_k, \end{aligned} \tag{9}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{l\times n}$, with initial condition x_0, state $x_k \in \mathbb{R}^n$, observed output $y_k \in \mathbb{R}^l$, and exogenous disturbance $\nu_k \in \mathbb{R}^m$, satisfying the constraint

$$\|\nu_k\| \le 1 \quad \text{for all } k = 0, 1, 2, \dots \tag{10}$$

Let the pair (A, B) is controllable and the pair (A, C) is observable.

We construct a filter described by the difference equation

$$\widehat{x}_{k+1} = A\widehat{x}_k + \mathcal{F}(y_k - C\widehat{x}_k), \quad \widehat{x}_0 = 0,$$

for the state estimate $\widehat{x}_k$ with a fixed matrix $\mathcal{F} \in \mathbb{R}^{n\times l}$.

Introduce the residual

$$\rho_k = x_k - \widehat{x}_k.$$

The problem is to find the matrix $\mathcal{F}$ that ensures the minimality of the invariant ellipsoid $\mathcal{V}$ containing the residual ρ_k.

The following theorem holds.

Theorem 3 ([14]): Let $\widehat{\Omega}$ and $\widehat{\Psi}$ be the solution of the optimization problem

$$\min \operatorname{tr} \Upsilon$$

under the constraints

$$\begin{pmatrix} -\mu\Omega & (\Omega A - \Psi C)^{\mathrm{T}} & 0 \\ \Omega A - \Psi C & -\Omega & \Omega B - \Psi D \\ 0 & (\Omega B - \Psi D)^{\mathrm{T}} & -(1-\mu)I \end{pmatrix} \preceq 0,$$

$$\begin{pmatrix} \Upsilon & I \\ I & \Omega \end{pmatrix} \succeq 0, \quad \Omega \succ 0,$$

in the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $0 < \mu < 1$.

Then the optimal filter matrix is given by the expression

$$\widehat{\mathcal{F}} = \widehat{\Omega}^{-1}\widehat{\Psi},$$

and the matrix of the minimal invariant ellipsoid for the residual ρ_k for system (9) with $x_0 = 0$ is given by the expression

$$\widehat{\Xi} = \widehat{\Omega}^{-1}.$$

The search procedure for a sparse solution of the filtering problem for system (9), (10) also involves performing three consecutive steps.

Algorithm 2:

Step 1. We solve the optimization problem

$$\min \operatorname{tr} \Upsilon \tag{11}$$

under the constraints

$$\begin{pmatrix} -\mu\Omega & (\Omega A - \Psi C)^{\mathrm{T}} & 0 \\ \Omega A - \Psi C & -\Omega & \Omega B - \Psi D \\ 0 & (\Omega B - \Psi D)^{\mathrm{T}} & -(1-\mu)I \end{pmatrix} \preceq 0, \tag{12}$$

$$\begin{pmatrix} \Upsilon & I \\ I & \Omega \end{pmatrix} \succeq 0, \quad \Omega \succ 0, \tag{13}$$

with the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar paramater $\mu > 0$.

The values Ω^*, Ψ^*, and Υ^* define the matrix

$$\mathcal{F}^* = (\Omega^*)^{-1}\Psi^*$$

of the optimal filter, the matrix

$$\Xi^* = (\Omega^*)^{-1}$$

of the minimal invariant ellipsoid for the residual, and the optimal value

$$\mathcal{J}^* = \operatorname{tr} \Upsilon^*$$

of the cost function.

Step 2. Having the value $\mathcal{J}^*$, we implement the relaxation coefficient $\lambda > 1$ and solve the optimization problem

$$\min \|\Psi\|_{c_1} \quad \text{s.t. (12), (13), and } \operatorname{tr} \Upsilon \preceq \lambda \mathcal{J}^*$$

in the matrix variables $\Omega \in \mathbb{S}^{n\times n}$, $\Psi \in \mathbb{R}^{n\times l}$, $\Upsilon \in \mathbb{S}^{n\times n}$, and the scalar parameter $0 < \mu < 1$. Due to the properties of the c_1-norm, one can expect the appearance of zero columns in its solution $\widehat{\Psi}_0$.

Step 3. We resolve the original problem (11)–(13) where the same arrangement of zero rows is fixed in the matrix variable Ψ as in the column-sparse matrix $\widehat{\Psi}_0$. Its solution $\widehat{\Omega}$, $\widehat{\Psi}$ defines the column-sparse filter matrix

$$\widehat{\mathcal{F}} = \widehat{\Omega}^{-1}\widehat{\Psi}$$

and the matrix

$$\widehat{\Xi} = \widehat{\Omega}^{-1}$$

of the corresponding invariant ellipsoid for the residual.

Remarks 1 and 2 remain valid in the discrete-time statements.

As the results of numerical simulations show, the "payment" for using a reduced number of controls/outputs (i.e., a loss in terms of cost function) is usually very small.

V. Example

Consider the HE3 problem borrowed from COMPleib [24] benchmark library. This library contains various problems having a clear engineering origin and using to test the efficacy of the proposed approaches. The considered linearized system describes the dynamics of the Bell201A-1 helicopter.

The matrices of the considered system have the following form:

$$A = \begin{pmatrix} -0.0046 & 0.038 & 0.3259 & -0.0045 & -0.402 & -0.073 & -9.81 & 0 \\ -0.1978 & -0.5667 & 0.357 & -0.0378 & -0.2149 & 0.5683 & 0 & 0 \\ 0.0039 & -0.0029 & -0.2947 & 0.007 & 0.2266 & 0.0148 & 0 & 0 \\ 0.0133 & -0.0014 & -0.4076 & -0.0654 & -0.4093 & 0.2674 & 0 & 9.81 \\ 0.0127 & -0.01 & -0.8152 & -0.0397 & -0.821 & 0.1442 & 0 & 0 \\ -0.0285 & -0.0232 & 0.1064 & 0.0709 & -0.2786 & -0.7396 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B=\begin{pmatrix}0.0676\\-1.1151\\0.0062\\-0.017\\-0.0129\\0.139\\0\\0\end{pmatrix},\qquad C=\begin{pmatrix}0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0\end{pmatrix},\qquad D=\begin{pmatrix}0\\0.1\\0\\0\\0.05\\0\end{pmatrix}$$

Here the state vector is given as

$$x=\begin{pmatrix}u_H & \sigma & h & v & p & r & \theta & \varphi\end{pmatrix}^{\mathrm{T}},$$

where u_H is the forward velocity, h is the pitch rate, σ is the vertical velocity, p is the roll rate, v is the lateral velocity, r is the yaw rate, φ is the roll angle, θ is the pitch angle, and the output vector is

$$y=\begin{pmatrix}\sigma & \theta & \varphi & r & h & p\end{pmatrix}^{\mathrm{T}}.$$

Setting $\Xi_0 = 0.1I$ and using Theorem 2, at the first step of Algorithm 1 we obtain the optimal filter matrix $\mathcal{F}^*$ and the corresponding invariant ellipsoid for the residual with matrix Ξ^* such that

$$\operatorname{tr}\Xi^* = 1.1381.$$

At the second step, solving the c_1-optimization problem (8) for $\lambda = 10$, we obtain the matrix $\widehat{\Psi}_0$ with two last columns of the order of 10^{-10}.

At the third step, fixing these rows as zero and resolving the original problem, we obtain the column-sparse filter matrix $\widehat{\mathcal{F}}$ and the invariant ellipsoid $\widehat{\Xi}$ for the residual with

$$\operatorname{tr}\widehat{\Xi} = 1.2131.$$

$$\mathcal{F}^*=\begin{pmatrix}-3.3724&-0.6504&0.0449&0.7496&1.8176&-0.6771\\1.2395&-10.5478&-0.0284&-0.2938&-0.9630&0.2334\\0.8538&-0.1075&-0.0023&-0.1830&-0.0243&-0.1172\\-0.0429&0.0287&9.8102&0.3401&-0.3943&-0.4499\\-0.3655&0.0872&1.0006&-0.1138&-0.4393&-0.5462\\0.2890&1.2176&0.0021&-0.4221&0.3213&-0.0226\\6.8280&-0.7092&-0.0062&-0.9332&0.7106&0.0687\\0.0073&-0.0000&0.2784&0.0000&0.0000&-0.0000\end{pmatrix}$$

$$\Xi^*=\begin{pmatrix}0.3714&-0.1278&-0.0139&0.0007&0.0036&0.0124&-0.0377&0.0000\\-0.1278&0.1602&0.0065&-0.0003&-0.0017&-0.0058&0.0177&-0.0000\\-0.0139&0.0065&0.1007&-0.0000&-0.0002&-0.0006&0.0019&-0.0000\\0.0007&-0.0003&-0.0000&0.1000&0.0000&0.0000&-0.0001&-0.0000\\0.0036&-0.0017&-0.0002&0.0000&0.1000&0.0002&-0.0005&0.0000\\0.0124&-0.0058&-0.0006&0.0000&0.0002&0.1006&-0.0017&0.0000\\-0.0377&0.0177&0.0019&-0.0001&-0.0005&-0.0017&0.1052&-0.0000\\0.0000&-0.0000&-0.0000&-0.0000&0.0000&0.0000&-0.0000&0.1000\end{pmatrix}$$

$$\widehat{\Psi}_0=\begin{pmatrix}-0.4093&-2.0441&0.1151&-0.1159&0&0\\0.4093&-2.0441&-0.2968&-0.2514&0&0\\0.4093&2.0441&-1.0477&-0.0129&0&0\\0.0724&-0.3055&1.9967&0.2514&0&0\\-0.4093&-0.3780&1.9967&-0.2514&0&0\\-0.4093&2.0441&1.1985&0.2514&0&0\\-0.2441&2.0441&1.9967&0.2514&0&0\\-0.0143&-0.4028&1.9967&0.0245&0&0\end{pmatrix}$$

$$\widehat{\mathcal{F}}=\begin{pmatrix}-1.4878&0.6754&0.0519&0.5895&0&0\\0.7782&-11.1508&-0.3270&-0.2482&0&0\\0.7283&0.0624&-0.0159&0.0733&0&0\\-0.8973&-0.1698&9.9423&0.4216&0&0\\-0.8263&-0.1289&1.0840&-0.0998&0&0\\0.3308&1.3900&0.0164&-0.4074&0&0\\8.0516&-0.0006&-0.5781&-0.9786&0&0\\0.1116&0.0000&0.3123&-0.0170&0&0\end{pmatrix}$$

$$\widehat{\Xi} = \begin{pmatrix} 0.4183 & -0.1314 & 0.0026 & -0.0146 & -0.0251 & 0.0179 & -0.0083 & -0.0147 \\ -0.1314 & 0.1571 & 0.0013 & 0.0007 & 0.0069 & -0.0076 & 0.0104 & 0.0050 \\ 0.0026 & 0.0013 & 0.1019 & -0.0044 & -0.0030 & -0.0000 & 0.0055 & -0.0010 \\ -0.0146 & 0.0007 & -0.0044 & 0.1105 & 0.0076 & -0.0004 & -0.0125 & 0.0026 \\ -0.0251 & 0.0069 & -0.0030 & 0.0076 & 0.1062 & -0.0011 & -0.0077 & 0.0024 \\ 0.0179 & -0.0076 & -0.0000 & -0.0004 & -0.0011 & 0.1010 & -0.0010 & -0.0007 \\ -0.0083 & 0.0104 & 0.0055 & -0.0125 & -0.0077 & -0.0010 & 0.1170 & -0.0022 \\ -0.0147 & 0.0050 & -0.0010 & 0.0026 & 0.0024 & -0.0007 & -0.0022 & 0.1011 \end{pmatrix}$$

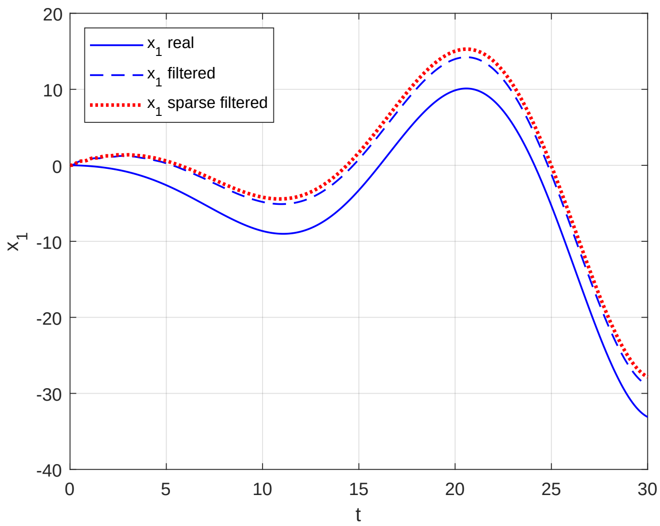

Fig. 1. Filtering the coordinate x_1.

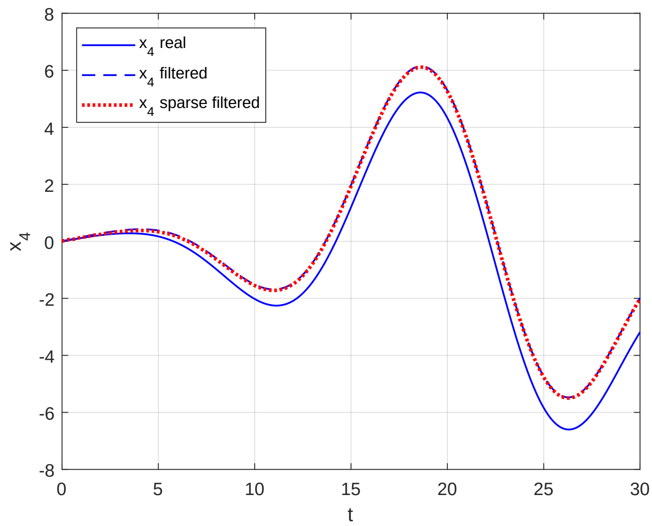

Fig. 2. Filtering the coordinate x_4.

Thus, we construct the sparse filter not using the outputs $y_5 = h$ (pitch rate) and $y_6 = p$ (roll rate), wherein the loss by the cost criterion is 6.5% only.

In the Fig. 1, the solid line depicts the trajectory $x_1(t) = u_H$ (forward velocity) of the system for some admissible exogenous disturbance, the dashed line depicts its optimal estimate $\widehat{x}_1(t)$, and the red dotted line depicts the result $\widetilde{x}_1(t)$ of using the proposed sparse filtering procedure.

For the coordinate $x_4 = v$ (lateral velocity), the sparse filtering accuracy is even higher, see Fig. 2.

The quality of filtering by other coordinates is also quite high.

From a computational point of view, the computations according to Algorithm 1 do not present any technical difficulties. At all its steps, we are dealing with convex optimization problems, for which the MATLAB-based cvx package mentioned in the introduction can be effectively used.

VI. Conclusion

We propose an approach to the sparse filtering problem under nonrandom bounded exogenous disturbances using an observer. The approach is based on the LMI technique and the method of invariant ellipsoids. Using of this concept made it possible to reduce the original problem to a semidefinite programming that can be easily solved numerically. The approach is simple and easily implementable; it equally covers both continuous- and discrete-time cases.

In the future, the author plans to expand the results obtained to the various robust formulations of the problem, in particular, to the system

$$\dot{x} = (A + F\Delta H)x + B\nu$$

subjected to norm-bounded matrix uncertainty $\Delta \in \mathbb{R}^{p\times q}$, $\|\Delta\| \le 1$, where F, H are given matrices of the appropriate dimensions.

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