Flat input based canonical form observers for non-integrable nonlinear systems

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***Abstract*—In this contribution, the design of canonical form observers for nonlinear non-integrable systems is investigated. These systems cannot be transformed into observer canonical form, therefore an exact observer error linearization cannot be achieved. However, using flat inputs and dynamic compensators, the original dynamics can be rendered into an integrable flat system. For this modified flat input system a canonical form observer can be designed. By utilizing a state transformation, it is then possible to obtain an estimate of the original state, where the observer error is approximately linearized. This procedure is exemplified by the Rössler system. Furthermore, we illustrate the relationship of this approach with high-gain observers.**

***Index Terms*—nonlinear systems, observer design, observer error linearization, differential flatness, flat inputs**

I. Introduction

Typically, the available measurements of a nonlinear process do not directly reveal the full state of the dynamical system. State estimation has therefore become an important topic from both a theoretical and an applied point of view. The corresponding tools to solve this problem are called observers and have attracted a lot of attention over the past decades. Due to the well understood linear case, for nonlinear systems typical observer designs are based on certain canonical forms. A rather popular choice is the canonical form observer (CFO) which is based on the observer canonical form (OCF) [1], where the dynamics are split into a trivial linear and a nontrivial nonlinear part. The nonlinearities are allowed to solely depend on the measurements and the input of the system. These nonlinearities are then compensated by the observer such that the error dynamics becomes linear [2], [3].

However, the conditions for the existence of a transformation into the OCF are rather restrictive which make this method unapplicable in many cases, the main issue being an integrability condition. There exist several ways to overcome these restrictions, such as

- a partial linearization of the observer error dynamics [4]–[6],
- using time scaling in the transformations [7],
- allowing the transformation into OCF to depend on the input and its time derivatives [8]–[10],
- embedding the system in a higher dimensional space [11], or by
- approximately linearizing the observer error dynamics [2], [12]–[18].

Differential flatness on the other hand, introduced in the early 1990s via the existence of so-called flat outputs [19], is a fundamental property for understanding dynamic feedback linearization, and plays a key role in many open and closed loop control methods. Despite its significance and a large research effort [20]–[25], to date rather important questions regarding the existence and the computation of flat outputs have not been answered. Motivated by the question of ideal actuator placement, flat inputs have been introduced as a dual concept to flat outputs [26], [27] and the computation has been investigated [28]–[31]. Although, flat inputs have turned out to be valuable for control problems [32], [33], too, naturally one would assume that they play a role in state estimation. Inspired by flat input based control methods, a Kalman filter approach to state estimation has been developed [34].

As an extension of the conference paper [35], in this article we describe a flat input based approach for approximate observer error linearization using the OCF. By using dynamic compensators, the original non-integrable system (OS) is altered such that the modified system (MS) is an integrable flat input system. That is, the modified system can be transformed into OCF, and the design of a standard CFO is possible which allows a state estimation of the modified system. In order to estimate the state of the original system, a state transformation is constructed. Unlike in flat input control schemes, where typically a dynamic compensator cannot be stated analytically and therefore requires approaches like discretization, the compensator used for the observer method as described here can always be given analytically. In order to find the modified system, one is rather free and could in principle use any suitable canonical form. However, this may lead to rather complicated computations, and instead, a modified system which is close to the original system from a structural point of view can simplify computations significantly.

The article is structured as follows: In Sect. II we introduce the necessary preliminaries, in Sect. III we describe the observer approach based on flat inputs and dynamic compensators. This method is demonstrated in Sect. IV on the Rössler system. The relation between this approach and the high-gain observer (HGO) is exemplified on the Lorenz

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system.

II. Preliminaries

We consider (locally) observable systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \tag{1a}$$
$$y = h(\mathbf{x}) \tag{1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}$, and the maps $\mathbf{f}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $h\colon \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth.

For a fixed input u, the *flow* $\varphi_t^{\boldsymbol{\xi}}(\mathbf{x}_0)$ corresponding to the vector field $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{f}(\mathbf{x}, u)$ at time t is the set of solutions $\mathbf{x}(t)$ of (1a) with $\mathbf{x}(0) = \mathbf{x}_0$, i.e., $\varphi_t^{\boldsymbol{\xi}}(\mathbf{x}_0) = \mathbf{x}(t)$.

The function $\mathrm{L}_{\mathbf{f}}\, h(\mathbf{x}) := \frac{\partial h}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, u)$ is called *Lie derivative* of h along $\mathbf{f}$, and repeated Lie derivatives are defined by $\mathrm{L}_{\mathbf{f}}^k\, h(\mathbf{x}) := \mathrm{L}_{\mathbf{f}}(\mathrm{L}_{\mathbf{f}}^{k-1}\, h(\mathbf{x}))$ for $k \geqslant 1$ with $\mathrm{L}_{\mathbf{f}}^0\, h(\mathbf{x}) := h(\mathbf{x})$.

Local observability of (1) in a domain of interest $\mathcal{M} \subset \mathbb{R}^n$ is given if the *observability matrix*

$$\mathbf{Q}(\mathbf{x}) := \frac{\partial}{\partial \mathbf{x}} \begin{pmatrix} h(\mathbf{x}) \\ \mathrm{L}_{\mathbf{f}}\, h(\mathbf{x}) \\ \vdots \\ \mathrm{L}_{\mathbf{f}}^{n-1}\, h(\mathbf{x}) \end{pmatrix}$$

is invertible for all $\mathbf{x} \in \mathcal{M}$.

System (1) has *relative degree* $r \leqslant n$ if for all $i \in \{0, 1, \ldots, r-1\}$ we have

$$\frac{\partial\, \mathrm{L}_{\mathbf{f}}^i\, h(\mathbf{x})}{\partial u} = 0, \qquad \text{and} \qquad \frac{\partial\, \mathrm{L}_{\mathbf{f}}^r\, h(\mathbf{x})}{\partial u} \neq 0.$$

For notational convenience, we denote the signals $(a, \dot{a}, \ldots, a^{(i)})$ by $a^{[i]}$.

System (1a) is called (*differentially*) *flat* [36] if there exists a (possibly fictitious) output y_{f} with maps $\rho, \xi_{\mathbf{x}}$ and ξ_u such that

1) $y_{\mathrm{f}} = \rho(\mathbf{x}, u^{[\alpha]}), \alpha < \infty$,
2) $\mathbf{x} = \xi_{\mathbf{x}}(y_{\mathrm{f}}^{[n-1]})$, and
3) $u = \xi_u(y_{\mathrm{f}}^{[n]})$.

A flat system has full relative degree $r = n$ w.r.t. a flat output y_{f}.

Instead of searching for a flat output y_{f} of (1a), for (locally) observable input-independent systems

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}}(\mathbf{x}), \qquad y = h(\mathbf{x})$$

we can similarly compute control vector fields $\mathbf{g}_{\mathrm{f}}(\mathbf{x})$ such that

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}}(\mathbf{x}) + \mathbf{g}_{\mathrm{f}}(\mathbf{x}) u_{\mathrm{f}}$$

becomes flat, and $y = h(\mathbf{x})$ is a flat output. Then, $\mathbf{g}_{\mathrm{f}}$ is called a *flat input vector field*, and u_{f} is called a *flat input* [26], [27]. Assuming $\det \mathbf{Q}(\mathbf{x}) \neq 0$, for single-output systems the flat input vector field can be obtained by

$$\mathbf{g}_{\mathrm{f}}(\mathbf{x}) = \alpha(\mathbf{x}) \mathbf{Q}^{-1}(\mathbf{x}) \mathbf{e}_n \tag{2}$$

where $\mathbf{e}_n$ is the n-th unit vector, and $\alpha(\mathbf{x}) \not\equiv 0$ can be chosen arbitrarily [26]. For the multi-output case, the computation of flat input vector fields is investigated in [28]–[30].

The *Lie bracket* of two vector fields $\mathbf{f}$ and $\mathbf{g}$ is defined as

$$[\mathbf{f}, \mathbf{g}] := \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \mathbf{g}.$$

Repeated Lie brackets are defined as

$$\mathrm{ad}_{\mathbf{f}}^k\, \mathbf{g} := [\mathbf{f}, \mathrm{ad}_{\mathbf{f}}^{k-1}\, \mathbf{g}], \quad k \geqslant 1$$

with $\mathrm{ad}_{\mathbf{f}}^0\, \mathbf{g} := \mathbf{g}$.

If there exists a diffeomorphism $\mathbf{z} = \mathbf{T}(\mathbf{x})$ with $\mathbf{x} = \mathbf{T}^{-1}(\mathbf{z}) =: \mathbf{S}(\mathbf{z})$ such that (1) is transformed into

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \boldsymbol{\alpha}(\mathbf{c}^\top \mathbf{z}, u) \tag{3a}$$
$$y = \gamma(\mathbf{c}^\top \mathbf{z}) \tag{3b}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}, \quad \mathbf{c}^\top = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{4}$$

then (3) is called *observer canonical form* (OCF) of (1) with an additional invertible output transformation γ. An observer design for (3) is then straight forward and has the structure

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}\hat{\mathbf{z}} + \boldsymbol{\alpha}(\gamma^{-1}(y), u) + \boldsymbol{\ell}(\gamma^{-1}(y) - \mathbf{c}^\top \hat{\mathbf{z}})$$
$$\hat{\mathbf{x}} = \mathbf{S}(\hat{\mathbf{z}})$$

with $\boldsymbol{\ell} \in \mathbb{R}^n$ which results in an observer error $\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}}$ such that

$$\dot{\tilde{\mathbf{z}}} = (\mathbf{A} - \boldsymbol{\ell}\mathbf{c}^\top)\tilde{\mathbf{z}}. \tag{5}$$

Due to observability of (3), $\boldsymbol{\ell}$ can be used to arbitrarily place the eigenvalues of $\mathbf{A} - \boldsymbol{\ell}\mathbf{c}^\top$, and therefore enforce asymptotic stability of $\tilde{\mathbf{z}} = \mathbf{0}$ in (5).

The so-called *starting vector* is defined by

$$\boldsymbol{\nu}(\mathbf{x}) := \beta(h(\mathbf{x})) \mathbf{Q}^{-1}(\mathbf{x}) \mathbf{e}_n$$

where $\mathbf{e}_n = (0, \ldots, 0, 1)^\top$ is the n-th unit vector, and the scalar function $\beta(h(\mathbf{x})) \neq 0$. Defining $\mathbf{F}(\mathbf{x}) := \mathbf{f}(\mathbf{x}, 0)$ and $\mathbf{G}(\mathbf{x}, u) := \mathbf{f}(\mathbf{x}, u) - \mathbf{F}(\mathbf{x})$, the existence condition for the OCF are then given as follows [37]:

Theorem 1: System (1) can be transformed into (3) in the neighborhood of $\mathbf{s} \in \mathcal{M}$ via a local diffeomorphism γ iff there exists a function β with $\beta(h(\mathbf{s})) \neq 0$ such that

1) $\operatorname{rank} \mathbf{Q}(\mathbf{s}) = n$
2) $[\mathrm{ad}_{-\mathbf{F}}^i\, \boldsymbol{\nu}, \mathrm{ad}_{-\mathbf{F}}^j\, \boldsymbol{\nu}](\mathbf{x}) = \mathbf{0} \quad \forall i, j \in \{0, 1, \ldots, n-1\}$
3) $[\mathbf{G}, \mathrm{ad}_{-\mathbf{F}}^i\, \boldsymbol{\nu}](\mathbf{x}, u) = \mathbf{0} \quad \forall i \in \{0, 1, \ldots, n-2\}$

are satisfied for all $\mathbf{x}$ in a neighborhood of $\mathbf{s}$ and all $u \in \mathcal{U}$. There exists a global diffeomorphism if the above conditions are fulfilled for $\mathcal{M} = \mathbb{R}^n$ and additionally

4) $\mathrm{ad}_{-\mathbf{F}}^i\, \boldsymbol{\nu}$ are complete vector fields $\forall i \in \{0, 1, \ldots, n-1\}$.

Condition 1 is a local observability condition where $\mathbf{Q}(\mathbf{x})$ is taken w.r.t. the uncontrolled dynamics $\mathbf{F}(\mathbf{x})$, i.e.,

$$\mathbf{Q}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{x}} (h(\mathbf{x}), \mathrm{L}_{\mathbf{F}}\, h(\mathbf{x}), \ldots, \mathrm{L}_{\mathbf{F}}^{n-1}\, h(\mathbf{x}))^\top. \tag{6}$$

Condition 2 is an integrability condition which oftentimes is not satisfied.

III. Observer design via dynamic compensation for SISO systems

Flat input based control utilizes dynamic compensators to render a non-flat system flat [32], [33], which then allows the usage of flatness based method. In general, this requires the solution of a differential equation which may be difficult (if not impossible), or alternatively use a discretized compensator. Here, we attempt to use dynamic compensators for the observer design as well. However, instead of solving for the original input, we solve for the flat input which is always possible analytically.

We assume that the original locally observable system is not integrable, i.e., it does not satisfy the conditions of Theorem 1. The first step is to modify the (uncontrolled) system such that these conditions are met. We then compute a flat input vector field for this modified integrable system which in conjunction with a corresponding dynamic compensator renders the original output flat, i.e., the relative degree is equal to the system dimension, see Fig. 1.

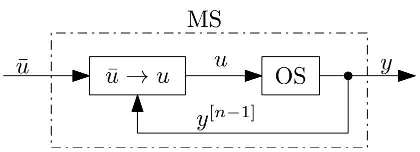

Fig. 1. Using dynamic input compensators to turn an original system into a modified system where $y^{[n-1]}$ denotes the feedback of $y, \dot{y}, \ldots, y^{(n-1)}$.

This results in a modified flat input system that can be transformed into OCF which allows the design of a CFO. Under certain conditions we can then compute the state components of the original system from the components of the state vector of the modified system. In order to design the observer, an additional dynamic compensator $(u \to \bar{u})$ needs to be attached to the input $\bar{u}$. The observer based on the modified system is shown in Fig. 2.

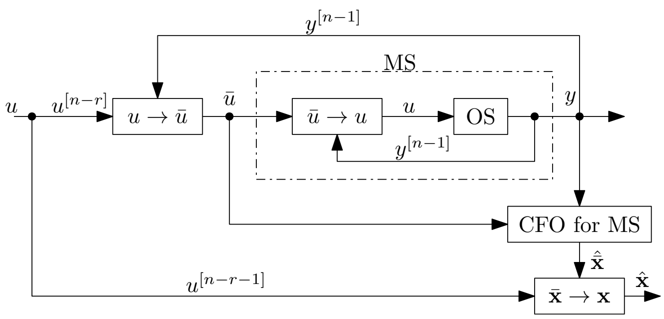

Fig. 2. Observer design based on dynamic input compensators $(\bar{u} \to u)$ where $\hat{\bar{\mathbf{x}}}$ and $\hat{\mathbf{x}}$ are observed variables.

However, this structure can be simplified as shown in Fig. 3 and has the advantage that the computation of $(\bar{u} \to u)$ is far easier than that of $(u \to \bar{u})$ due to full relative degree of the modified dynamics w.r.t. y.

Lastly, in order to avoid feedback of derivatives of the output y, we substitute $y^{(i)}$ in the compensator with iterated Lie derivatives of the output map of the modified system w.r.t. the observed state of the modified system, see Fig. 4.

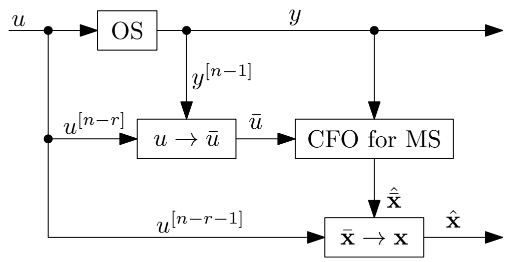

Fig. 3. Simplified observer design using $(u \to \bar{u})$.

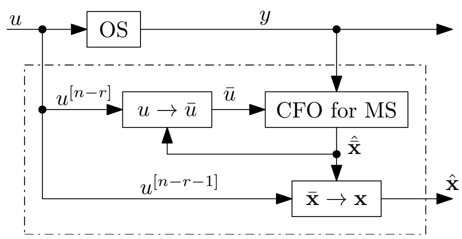

Fig. 4. Final observer design with dynamic compensation using Lie derivatives to avoid the feedback of output derivatives.

More precisely, we investigate (locally) observable systems of the form (1) with relative degree $r \leqslant \dim(\mathbf{x}) =: n$ w.r.t. (1b), and such that the integrability conditions of Theorem 1 are not satisfied, i.e., there exists no transformation of (1) into OCF. Furthermore, we assume that the input u and its derivatives $\dot{u}, \ldots, u^{(n-r)}$ are known at any time.

A. Modification of the original dynamics using flat inputs

After determining the problematic terms in (1) w.r.t. the integrability conditions in Theorem 1, in order to find a suitable modified system we first alter the uncontrolled right-hand side $\mathbf{f}(\mathbf{x}, 0)$ which gives an integrable $\bar{\mathbf{f}}(\bar{\mathbf{x}})$ (cf. [17]). Next, we compute a flat input vector field $\bar{\mathbf{g}}(\bar{\mathbf{x}})$ of $\bar{\mathbf{f}}(\bar{\mathbf{x}})$ such that the original output becomes flat. The modified flat system[1] then reads

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{f}}(\bar{\mathbf{x}}) + \bar{\mathbf{g}}(\bar{\mathbf{x}})\bar{u} \tag{7a}$$
$$\bar{y} = h(\bar{\mathbf{x}}) \tag{7b}$$

with $\dim(\bar{\mathbf{x}}) = n$ where $\bar{u}$ is a flat input, or equivalently, $\bar{y}$ is a flat output of (7a), i.e., (7a) has relative degree n w.r.t. $\bar{y}$. Note, that this implies (local) observability of (7). Hereinafter, we assume that (7) can be found such that the conditions of Theorem 1 are fulfilled.

In general, the closer the modified system is to the original system from a structural perspective, the easier the computations later.

[1]For clarification we denote the output of the modified system by $\bar{y}$ although this corresponds to the same measured output y, cf. Fig. 1.

B. Constructing the dynamic input compensator from input-output descriptions

Unlike in flat-input control as described in [32] or [33] where a compensation $(\bar{u} \to u)$ is needed, here, we only need the compensator that transforms the real input u into the input of the modified system $\bar{u}$, i.e., $(u \to \bar{u})$. This is easier since our modified system is assumed to have relative degree n. Due to observability and sufficient smoothness of both, (1) and (7), the input-output representations are equivalent to their state-space representations [38]. They can be determined by successivly differentiating the output and eliminating the state components. System (1) then has the form

$$y^{(n)} = p(y^{[n-1]}, u^{[n-r]}) \tag{8}$$

whereas due to full relative degree of system (7) we get

$$\bar{y}^{(n)} = \bar{q}(\bar{y}^{[n-1]}) + \bar{p}(\bar{y}^{[n-1]})\bar{u}. \tag{9}$$

Enforcing the input output behavior of (1) and (7) to be the same by equating (8) and (9) results in the dynamic compensator $(u \to \bar{u})$

$$\bar{u} = \frac{p(\cdot) - \bar{q}(\cdot)}{\bar{p}(\cdot)} =: \Xi(y^{[n-1]}, \bar{y}^{[n-1]}, u^{[n-r]}). \tag{10}$$

Note that for all non-negative integers i we have

$$y^{(i)} \overset{!}{=} \bar{y}^{(i)}, \tag{11}$$

i.e., (10) becomes

$$\bar{u} = \Xi(y^{[n-1]}, u^{[n-r]}). \tag{12}$$

System (7) extended by (12) now has the same input-output behavior as (1). Finally, in order to avoid the feedback of derivatives of y, we substitute $y^{(i)}$ with $\mathrm{L}_{\bar{\mathbf{f}}}^{i} h(\hat{\bar{\mathbf{x}}})$ in (12) which yields

$$\bar{u} = \Xi(h(\bar{\mathbf{x}}), \mathrm{L}_{\bar{\mathbf{f}}} h(\hat{\bar{\mathbf{x}}}), \ldots, \mathrm{L}_{\bar{\mathbf{f}}}^{n-1} h(\hat{\bar{\mathbf{x}}}), u^{[n-r]}) = \tilde{\Xi}(\hat{\bar{\mathbf{x}}}, u^{[n-r]}). \tag{13}$$

C. CFO design for the modified system

The design of the CFO for the modified system (7) follows the standard approach [2], e.g., by computing a starting vector $\boldsymbol{\nu}(\bar{\mathbf{x}})$ and

$$\mathrm{ad}_{-\bar{\mathbf{f}}}^{i} \boldsymbol{\nu}(\bar{\mathbf{x}})$$

for $i = 0, 1, \ldots, n-1$. The transformation of (7) into OCF can then be obtained from the composition of the flows

$$\bar{\mathbf{x}} = \mathbf{S}(\mathbf{z}) = \varphi_{z_1}^{\boldsymbol{\nu}} \circ \varphi_{z_2}^{\mathrm{ad}_{-\bar{\mathbf{f}}} \boldsymbol{\nu}} \circ \cdots \circ \varphi_{z_n}^{\mathrm{ad}_{-\bar{\mathbf{f}}}^{n-1} \boldsymbol{\nu}}(\mathbf{p}). \tag{14}$$

D. State transformation

The CFO as designed in the previous step allows the observation of the state vector $\bar{\mathbf{x}}$ of the system (7). In order to estimate $\mathbf{x}$ we need a conversion $(\bar{\mathbf{x}} \to \mathbf{x})$.

Assuming (1b) to have relative degree $r \leqslant n$ means for all $i \in \{0, 1, \ldots, r-1\}$ we have

$$\frac{\partial \mathrm{L}_{\mathbf{f}}^{i} h(\mathbf{x})}{\partial u} = 0, \qquad \text{and} \qquad \frac{\partial \mathrm{L}_{\mathbf{f}}^{r} h(\mathbf{x})}{\partial u} \neq 0,$$

i.e.,

$$\begin{aligned} y = \Phi_1(\mathbf{x}) \quad &= h(\mathbf{x}) \\ \dot{y} = \Phi_2(\mathbf{x}) \quad &= \mathrm{L}_{\mathbf{f}} h(\mathbf{x}) \\ \vdots \qquad & \quad \vdots \\ y^{(r)} = \Phi_{r+1}(\mathbf{x}, u) &= \mathrm{L}_{\mathbf{f}}^{r} h(\mathbf{x}). \end{aligned}$$

The subsequent $n-r$ derivatives then depend on derivatives of u and read

$$\begin{aligned} y^{(r+1)} &= \Phi_{r+2}(\mathbf{x}, u^{[1]}) \\ &\vdots \\ y^{(n-1)} &= \Phi_n(\mathbf{x}, u^{[n-r-1]}). \end{aligned}$$

For fixed $u, \dot{u}, \ldots, u^{(n-r-1)}$ the map $\boldsymbol{\Phi} := (\Phi_1, \ldots, \Phi_n)^\top$ is a local diffeomorphism, i.e., locally there is an inverse map

$$\mathbf{x} = \boldsymbol{\Phi}^{-1}(y^{[n-1]}, u^{[n-r-1]}). \tag{15}$$

For (7) we get

$$\begin{aligned} \bar{y} &= h(\bar{\mathbf{x}}) \\ \dot{\bar{y}} &= \mathrm{L}_{\bar{\mathbf{f}}} h(\bar{\mathbf{x}}) \\ &\vdots \\ \bar{y}^{(n-1)} &= \mathrm{L}_{\bar{\mathbf{f}}}^{n-1} h(\bar{\mathbf{x}}) \\ \bar{y}^{(n)} &= \mathrm{L}_{\bar{\mathbf{f}}}^{n} h(\bar{\mathbf{x}}) + \mathrm{L}_{\bar{\mathbf{g}}} \mathrm{L}_{\bar{\mathbf{f}}}^{n} h(\bar{\mathbf{x}}) \cdot \bar{u} \end{aligned} \tag{16}$$

with $\mathrm{L}_{\bar{\mathbf{g}}} \mathrm{L}_{\bar{\mathbf{f}}}^{n} h(\bar{\mathbf{x}}) \neq 0$. Finally, using (11) and (16) turns (15) into

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\Phi}^{-1}(h(\bar{\mathbf{x}}), \mathrm{L}_{\bar{\mathbf{f}}} h(\bar{\mathbf{x}}), \ldots, \mathrm{L}_{\bar{\mathbf{f}}}^{n-1} h(\bar{\mathbf{x}}), u^{[n-r-1]}) \\ &=: \boldsymbol{\Psi}(\bar{\mathbf{x}}, u^{[n-r-1]}). \end{aligned} \tag{17}$$

Note that we have $\mathbf{x} = \boldsymbol{\Psi}(\bar{\mathbf{x}})$ if system (1a) has full relative degree w.r.t. the output (1b).

We now have the building blocks for our observer method.

Remark 1: From Section III we can deduce the autonomous case by setting $u^{[i]} = 0$, which results in the observer structure shown in Fig. 5. The key steps are the same, see IV-B for an example.

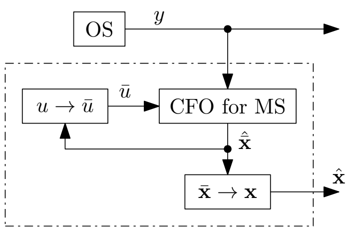

Fig. 5. Observer design for autonomous single-output system.

Remark 2: Even if $\boldsymbol{\Phi}$ is locally invertible, there may not exist a closed formula for (15). In this case, an additional output transformation may help, see Fig. 6 for the resulting observer structure, and [35] for an example.

Remark 3: The observer approach as described above yields stable approximately linear error dynamics, which can be

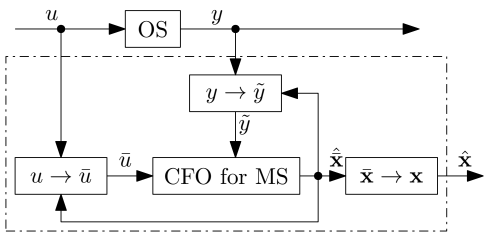

Fig. 6. Observer structure with an additional output transformation.

interpreted as a low pass filter that supresses sensor noise. The pole placement can also be done using typical filter design methods, such as the Butterworth- or Bessel-filter [45], [46].

IV. Examples

A. Rössler attractor

We consider the Rössler system [39], [40]

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_2 - x_3 \\ x_1 + a x_2 \\ c + x_3(x_1 - b) \end{pmatrix} \tag{18a}$$

$$y = h(\mathbf{x}) = x_1 \tag{18b}$$

with bifurcation parameters $a, b, c > 0$. A detailed observability analysis of this system is carried out in [41], [42]. The first Lie derivatives of (18a) w.r.t. y yield

$$\dot{y} = -x_2 - x_3 \tag{19a}$$

$$\ddot{y} = -x_1 - a x_2 - c - x_3(x_1 - b) \tag{19b}$$

$$\dddot{y} = -a(a x_2 + x_1) + x_2 + x_3(x_2 + x_3) + x_3 + (b - x_1)(c - x_3(b - x_1)). \tag{19c}$$

The observability matrix then reads

$$\mathbf{Q}(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -x_3 - 1 & -a & b - x_1 \end{pmatrix} \tag{20}$$

with $\det \mathbf{Q}(\mathbf{x}) = x_1 - a - b$, i.e., $x_1 \neq a + b$. Solving (19) for x_i we get

$$x_1 = y \tag{21a}$$

$$x_2 = \frac{y\dot{y} - b\dot{y} - c - y - \ddot{y}}{a + b - y} \tag{21b}$$

$$x_3 = \frac{c + y + \ddot{y} - a\dot{y}}{a + b - y} \tag{21c}$$

Computing the input-output representation results in

$$\begin{aligned} \dddot{y} = \frac{1}{y - a - b} &\left\{ a^2(-b\dot{y} - c + y\dot{y} - \ddot{y}) \right. \\ &+ a(-b^2\dot{y} - bc + 2by\dot{y} + by + cy - y^2\dot{y} - y^2 - \dot{y}^2 + \dot{y}) \\ &+ b^2 y + b^2\ddot{y} - 2by^2 - 2by\ddot{y} + b\dot{y} \\ &\left. + c\dot{y} + y^3 + y^2\ddot{y} + \ddot{y}\dot{y} \right\}. \end{aligned} \tag{22}$$

It is rather easy to check that the problematic term in $\mathbf{f}$ with regards to integrability is the nonlinearity $x_3(x_1 - b)$ in the third equation of (18a). Modifying this term to $x_1 - b$ yields the linear integrable system

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{f}}(\bar{\mathbf{x}}) = \begin{pmatrix} -\bar{x}_2 - \bar{x}_3 \\ \bar{x}_1 + a\bar{x}_2 \\ \bar{x}_1 + c - b \end{pmatrix} \tag{23a}$$

$$y = \bar{h}(\bar{\mathbf{x}}) = \bar{x}_1 \tag{23b}$$

The first Lie derivatives of (23a) w.r.t. (23b) read

$$\dot{y} = -\bar{x}_2 - \bar{x}_3 \tag{24a}$$

$$\ddot{y} = -a\bar{x}_2 + b - c - 2\bar{x}_1 \tag{24b}$$

The corresponding observability matrix is

$$\mathbf{Q}(\bar{\mathbf{x}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -2 & -a & 0 \end{pmatrix} \tag{25}$$

where $\det \mathbf{Q}(\bar{\mathbf{x}}) = -a$. We compute a flat input according to (2) and get

$$\bar{\mathbf{g}}(\bar{\mathbf{x}}) = \alpha(\bar{\mathbf{x}})\mathbf{Q}^{-1}(\bar{\mathbf{x}})\mathbf{e}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \tag{26}$$

where we choose $\alpha(\bar{\mathbf{x}}) = a$. The integrable modified flat input system therefore results in

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{f}}(\bar{\mathbf{x}}) + \bar{\mathbf{g}}(\bar{\mathbf{x}})\bar{u} = \begin{pmatrix} -\bar{x}_2 - \bar{x}_3 \\ \bar{x}_1 + a\bar{x}_2 \\ \bar{x}_1 + c - b \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \bar{u} \tag{27a}$$

$$y = \bar{h}(\bar{\mathbf{x}}) = \bar{x}_1 \tag{27b}$$

The first Lie derivatives of (27) yield

$$\dot{y} = -\bar{x}_2 - \bar{x}_3 \tag{28a}$$

$$\ddot{y} = -2\bar{x}_1 - a\bar{x}_2 + b - c \tag{28b}$$

$$\dddot{y} = -a\bar{x}_1 + (2 - a^2)\bar{x}_2 + 2\bar{x}_3 + a\bar{u} \tag{28c}$$

Solving for $\bar{x}_i$ we get

$$\bar{x}_1 = y \tag{29a}$$

$$\bar{x}_2 = \frac{b - c - 2y - \ddot{y}}{a} \tag{29b}$$

$$\bar{x}_3 = \frac{c + 2y + \ddot{y} - b}{a} - \dot{y}. \tag{29c}$$

The input-output representation of (27) reads

$$\dddot{y} = a(c - b + y + \bar{u} + \ddot{y}) - 2\dot{y} \tag{30}$$

We choose the starting vector

$$\boldsymbol{\nu} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \tag{31}$$

and compute the iterated Lie brackets

$$\mathrm{ad}_{-\bar{\mathbf{f}}}\,\boldsymbol{\nu} = \begin{pmatrix} 0 \\ -a \\ 0 \end{pmatrix} \ \mathrm{ad}^2_{-\bar{\mathbf{f}}}\,\boldsymbol{\nu} = \begin{pmatrix} a \\ -a^2 \\ 0 \end{pmatrix}. \tag{32}$$

The corresponding flows can be computed as

$$\varphi_{z_1}^{\boldsymbol{\nu}}(\bar{\mathbf{x}}) = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 - z_1 \\ \bar{x}_3 + z_1 \end{pmatrix}, \qquad \varphi_{z_2}^{\mathrm{ad}_{-\bar{\mathbf{f}}}\boldsymbol{\nu}}(\bar{\mathbf{x}}) = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 - az_2 \\ \bar{x}_3 \end{pmatrix}$$

$$\text{and} \quad \varphi_{z_3}^{\mathrm{ad}^2_{-\bar{\mathbf{f}}}\boldsymbol{\nu}}(\bar{\mathbf{x}}) = \begin{pmatrix} \bar{x}_1 + az_3 \\ \bar{x}_2 - a^2 z_3 \\ \bar{x}_3 \end{pmatrix}.$$

The transformation into OCF with $\mathbf{p} = (0,0,0)^\top$ can be calculated by concatenation of these flows and yields

$$\bar{\mathbf{x}} = \mathbf{S}(\mathbf{z}) = \varphi_{z_1}^{\boldsymbol{\nu}} \circ \varphi_{z_2}^{\mathrm{ad}_{-\bar{\mathbf{f}}}\boldsymbol{\nu}} \circ \varphi_{z_3}^{\mathrm{ad}^2_{-\bar{\mathbf{f}}}\boldsymbol{\nu}}(\mathbf{p}) = \begin{pmatrix} az_3 \\ -z_1 - az_2 - a^2 z_3 \\ z_1 \end{pmatrix} \tag{33}$$

The inverse map reads

$$\mathbf{z} = \mathbf{T}(\bar{\mathbf{x}}) = \begin{pmatrix} \bar{x}_3 \\ -\frac{\bar{x}_3}{a} - \bar{x}_1 - \frac{\bar{x}_2}{a} \\ \frac{\bar{x}_1}{a} \end{pmatrix} \tag{34}$$

and leads to the transformed system

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{z} + \begin{pmatrix} az_3 + c - b \\ -2z_3 + c - b \\ az_3 \end{pmatrix}. \tag{35}$$

As a result we get the CFO

$$\dot{\hat{\mathbf{z}}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \hat{\mathbf{z}} + \begin{pmatrix} a\hat{z}_3 + c - b \\ -2\hat{z}_3 + c - b \\ a\hat{z}_3 \end{pmatrix} + \mathbf{L}(y - a\hat{z}_3). \tag{36}$$

Computing the dynamic compensator gives

$$\begin{aligned} \bar{u} = {} & \frac{1}{a(a+b-\bar{x}_1)} \{a^2(b(1-\bar{x}_3) + \bar{x}_1\bar{x}_3 - \bar{x}_1) \\ & -a(\bar{x}_3[b^2 - 2b\bar{x}_1 + \bar{x}_1^2 - \bar{x}_2 - \bar{x}_3 + 1] - bc + c\bar{x}_1 + \bar{x}_2) \\ & -b^3 + b^2 c + 3b^2\bar{x}_1 - 2bc\bar{x}_1 - 3b\bar{x}_1^2 + c\bar{x}_1^2 + \bar{x}_1^3\}, \end{aligned} \tag{37}$$

and, finally, the state transformation yields

$$x_1 = \bar{x}_1 \tag{38a}$$

$$x_2 = \frac{a\bar{x}_2 - b + (\bar{x}_2 + \bar{x}_3)(b - \bar{x}_1) + \bar{x}_1}{a + b - \bar{x}_1} \tag{38b}$$

$$x_3 = \frac{-a\bar{x}_2 + a(\bar{x}_2 + \bar{x}_3) + b - \bar{x}_1}{a + b - \bar{x}_1} \tag{38c}$$

Simulation results in Fig.7.

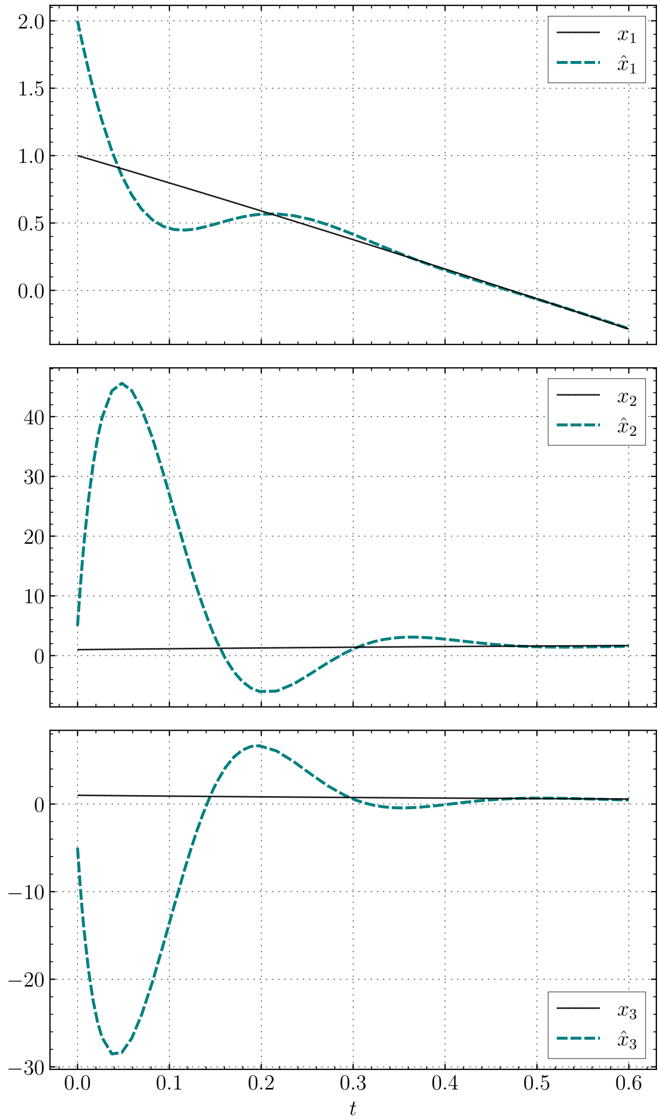

Fig. 7. Simulation results of the Rössler system with parameters $a = 0.55, b = 4, c = 2$, and the flat input based CFO with dynamic compensation. The initial conditions are $\mathbf{x}(0) = (1,1,1)^\top$ and $\hat{\mathbf{x}}(0) = (2,5,-5)^\top$. The eigenvalues are placed at -20.

B. The relation to high-gain observers

The procedure above has been shown to result in an observer design that can be utilized for non-integrable systems to achieve approximately linearized error dynamics. The main drawback of this method is the typically rather complicated computation of the OCF and the corresponding transformations.

Instead of modifying the original non-integrable dynamics such that integrability is achieved, and a subsequent flat input computation, one may instead use the fact that any flat state space system can be transformed into Brunovský canonical form. Designing an observer is then rather easy due to absent nonlinearities. Therefore, one may be tempted render the original dynamics into any flat input system (regardless of its integrability), to compute a dynamic compensator and a state transformation, but replace the CFO in Fig. 3 with an observer based on the Brunovský canonical form of the flat input system. It turns out that the resulting observer scheme is in fact a HGO.

This can easily be seen using an example: In [35], a flat input based CFO for the Lorenz system [43]

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} s(x_2 - x_1) \\ \rho x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - b x_3 \end{pmatrix} \tag{39}$$

with $y = h(\mathbf{x}) = x_1$ and the parameters $s, \rho, b > 0$ has been computed (cf. [44]). For that, the modified integrable flat input system

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{f}}(\bar{\mathbf{x}}) = \begin{pmatrix} s(\bar{x}_2 - \bar{x}_1) \\ \rho\bar{x}_1 - \bar{x}_2 - \bar{x}_3 \\ \bar{x}_1\bar{x}_2 - b\bar{x}_3 + \bar{u} \end{pmatrix} \tag{40}$$

with $\bar{y} = h(\bar{\mathbf{x}}) = \bar{x}_1$ was used. The corresponding dynamic compensator is

$$\bar{u} = \bar{x}_1\bar{x}_2(\bar{x}_1 - 1) + s\bar{x}_3(\tfrac{\bar{x}_2}{\bar{x}_1} - 1), \tag{41}$$

and the state transformation with its inverse

$$\mathbf{x} = \mathbf{T}(\bar{\mathbf{x}}) = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \frac{\bar{x}_3}{\bar{x}_1} \end{pmatrix}, \qquad \bar{\mathbf{x}} = \mathbf{S}(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ x_1x_3 \end{pmatrix}. \tag{42}$$

Instead of using a CFO for system (40), here, we will base the observer on the Brunovský canonical form. The observability map of (40)

$$\mathbf{z} := \begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \end{pmatrix} = \bar{\mathbf{q}}(\bar{\mathbf{x}}) = \begin{pmatrix} \bar{x}_1 \\ s(\bar{x}_2 - \bar{x}_1) \\ s(\rho\bar{x}_1 - \bar{x}_2 - \bar{x}_3 + s(\bar{x}_1 - \bar{x}_2)) \end{pmatrix} \tag{43}$$

in conjunction with the input transformation

$$\begin{aligned} v := b\rho y - by - b\dot{y} - \frac{b\ddot{y}}{s} - \frac{b\dot{y}}{s} + \rho\dot{y} \\ - y^2 - \ddot{y} - \dot{y} - \frac{y\dot{y}}{s} - \frac{\dddot{y}}{s} - \frac{\ddot{y}}{s} \end{aligned} \tag{44}$$

transforms the modified flat input system into Brunovský canonical form

$$\dot{\mathbf{z}} = \begin{pmatrix} z_2 \\ z_3 \\ v \end{pmatrix}, \qquad y = z_1. \tag{45}$$

An observer for the system (40) based on the linear system (45) then reads

$$\dot{\hat{\mathbf{z}}} = \begin{pmatrix} \hat{z}_2 \\ \hat{z}_3 \\ v \end{pmatrix} + \boldsymbol{\ell}(y - \hat{z}_1). \tag{46}$$

In coordinates of (40) this observer yields

$$\begin{aligned} \dot{\hat{\bar{\mathbf{x}}}} &= \hat{\mathbf{f}}(\hat{\bar{\mathbf{x}}}) + \mathbf{k}_\infty(\boldsymbol{\ell}) \cdot (y - h(\hat{\bar{\mathbf{x}}})) \qquad (47) \\ &= \begin{pmatrix} s(\hat{\bar{x}}_2 - \hat{\bar{x}}_2) \\ \rho\hat{\bar{x}}_1 - \hat{\bar{x}}_2 - \hat{\bar{x}}_3 \\ \hat{\bar{x}}_1\hat{\bar{x}}_2 - b\hat{\bar{x}}_3 + \bar{u} \end{pmatrix} \\ &\quad + \begin{pmatrix} \ell_1 \\ \ell_1 + \ell_2\frac{1}{s} \\ \ell_1(\rho - 1) - \ell_2(1 + \frac{1}{s}) - \ell_3\frac{1}{s} \end{pmatrix}(y - \hat{\bar{x}}_1). \qquad (48) \end{aligned}$$

Inserting the dynamic compensator (41) and the state transformation (42) into (47) results in the observer in coordinates of the original system:

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \frac{\partial \mathbf{T}}{\partial \hat{\bar{\mathbf{x}}}}(\mathbf{S}(\hat{\mathbf{x}})) \cdot \dot{\hat{\bar{\mathbf{x}}}}\Big|_{\hat{\bar{\mathbf{x}}} = \mathbf{S}(\hat{\mathbf{x}})} \\ &= \begin{pmatrix} s(\hat{x}_2 - \hat{x}_1) \\ \rho\hat{x}_1 - \hat{x}_2 - \hat{x}_1\hat{x}_3 \\ \hat{x}_1\hat{x}_2 - b\hat{x}_3 \end{pmatrix} \\ &\quad + \begin{pmatrix} \ell_1 \\ \ell_1 + \ell_2\frac{1}{s} \\ \ell_1\frac{\rho - 1 - \hat{x}_3}{\hat{x}_1} - \ell_2\frac{s+1}{s\hat{x}_1} - \ell_3\frac{1}{s\hat{x}_1} \end{pmatrix}(y - \hat{x}_1). \end{aligned} \tag{49}$$

On the other hand, the observability matrix of the original system reads

$$\mathbf{Q}(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ -s & s & 0 \\ s(\rho + s - x_3) & -s(s+1) & -sx_1 \end{pmatrix}, \tag{50}$$

and its inverse yields

$$\mathbf{Q}^{-1}(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{s} & 0 \\ \frac{\rho - x_3 - 1}{x_1} & -\frac{s+1}{sx_1} & -\frac{1}{sx_1} \end{pmatrix}. \tag{51}$$

A HGO for the original system (39) therefore reads

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{Q}^{-1}(\hat{\mathbf{x}})\boldsymbol{\ell}(y - h(\hat{\mathbf{x}})) \\ &= \begin{pmatrix} s(\hat{x}_2 - \hat{x}_1) \\ \rho\hat{x}_1 - \hat{x}_2 - \hat{x}_1\hat{x}_3 \\ \hat{x}_1\hat{x}_2 - b\hat{x}_3 \end{pmatrix} \\ &\quad + \begin{pmatrix} \ell_1 \\ \ell_1 + \ell_2\frac{1}{s} \\ \ell_1\frac{\rho - \hat{x}_3 - 1}{\hat{x}_1} - \ell_2\frac{s+1}{s\hat{x}_1} - \ell_3\frac{1}{s\hat{x}_1} \end{pmatrix}(y - \hat{x}_1). \end{aligned} \tag{52}$$

Obviously, this is identical to the observer (49). The comparison of the HGO and the flat input based CFO from [35] is shown in Fig. 8 and the norms of the observer errors in Fig. 9.

V. Conclusion

In this paper we have investigated the problem of observer design for nonlinear non-integrable system, i.e., systems that cannot be transformed into observer canonical form. Altering the (uncontrolled) dynamics such that one obtains an integrable system with a subsequent flat input computation results in a flat system that can be transformed into observer canonical form, and therefore, a canonical form observer can be designed. In order to estimate the state components of the original dynamics, the observer of the modified flat system in conjunction with a dynamic compensator and a state transformation is utilized and thus enables approximate observer error linearization of non-integrable systems. This approach is exemplified using the well-known Rössler system.

However, even for integrable systems, the design of the canonical form observer is rather complicated computationally. Since our approach exploits a flat intermediate system, it is rather obvious to replace the observer canonical form with the Brunovský canonical form. Using the example of the Lorenz

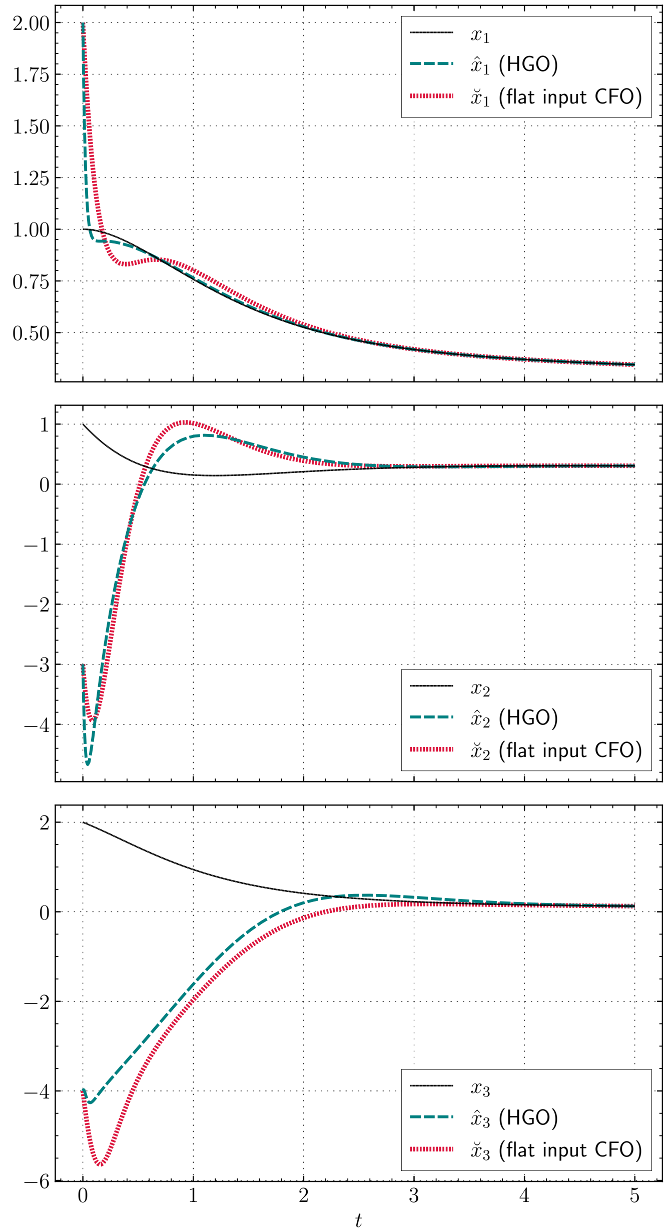

Fig. 8. Numerical simulation of the Lorenz system using initial data $\mathbf{x}(0) = (1, 1, 2)^\top$ and parameters $s = 0.5, \rho = 1, b = 1$, the HGO with initial data $\hat{\mathbf{x}}(0) = (2, -3, -4)^\top$ in comparison with the flat input CFO from [35] with initial data $\breve{\mathbf{x}}(0) = (2, -3, -4)^\top$. The eigenvalues of the observer error linearizations are placed at $-3, -3$ and -5.

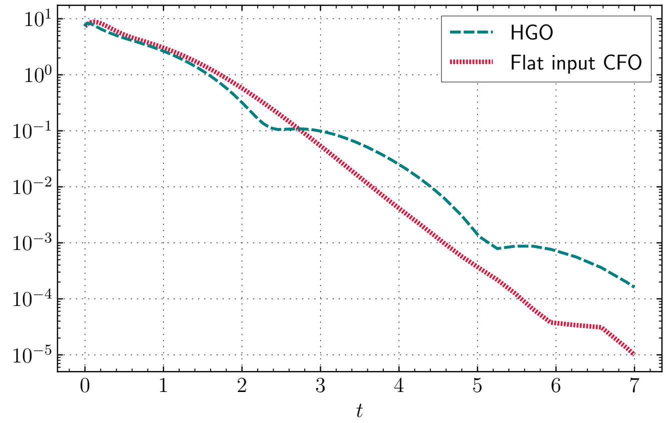

Fig. 9. Norm of the observer error for both, the HGO and the flat input CFO from [35] for the Lorenz system.

system, we show that the resulting observer corresponds to the high-gain observer, relating the flat input based canonical form observer with the high-gain observer. A comparison between both approaches is shown using simulations.

Practical implementation of the discussed flat input observer method is subject of future research.

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