SYSTEM THEORY, CONTROL AND COMPUTING JOURNAL, VOL. 2, NO. 1, JUNE 2022, pp. 22-33

# Full and Partial Eigenvalue Placement for Minimum Norm Static Output Feedback Control

Klaus Röbenack
*Institute of Control Theory*
*TU Dresden*
Dresden, Germany
klaus.roebenack@tu-dresden.de
ORCID 0000-0002-3347-0864

Daniel Gerbet
*Institute of Control Theory*
*TU Dresden*
Dresden, Germany
daniel.gerbet1@tu-dresden.de
ORCID 0000-0002-8527-8727

***Abstract*—The controller design for linear time-invariant state space systems seems to be straightforward and well established. This is not true for static output feedback control, which is still a challenging task. This paper deals with controller design based on eigenvalue assignment. We consider the placement of distinct as well as multiple real eigenvalues or complex conjugate pairs. The desired eigenvalue configurations are characterized in terms of algebraic divisibility of the characteristic polynomial of the closed-loop system. We also consider the problem of partial eigenvalue placement, where not all eigenvalues are fixed by feedback. Degrees of freedom in the controller design are used for the minimization of various matrix norms of the feedback gain matrix.**

***Index Terms*—Linear time-invariant systems, eigenvalue placement, static output feedback, polynomial ideals, Gröbner bases, quantifier elimination, norm minimization**

## I. Introduction

This paper deals with a controller design problem for linear time-invariant systems. We consider linear state space systems in the form

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

with matrices $A \in \mathbb{Q}^{n\times n}$, $B \in \mathbb{Q}^{n\times m}$, $C \in \mathbb{Q}^{r\times n}$ over the rational field $\mathbb{Q}$. The restriction to rational number allows an exact computer representation of the system.

Typically, controller design for system (1) is carried out by means of a state feedback. The feedback law is then implemented in combination with a state observer resulting in a dynamic output feedback control law. From the viewpoint of implementation, a static output feedback law

$$u = -Ky \tag{2}$$

with the gain matrix $K \in \mathbb{R}^{m\times r}$ would be advantageous. However, for both eigenvalue placement as well as stabilization, these problems are significantly more challenging compared to the state feedback control [1]–[6].

This paper extends the conference paper [7] presented at ICSTCC 2021. In [7], we considered the computation of a gain matrix $K$ of the feedback law (2) minimizing the Frobenius and the maximum norm, respectively. In this contribution we extended these approaches to the minimization of the spectral norm as well as the maximum absolute row and column sum norms.

In some applications, the full placement of all eigenvalues is not possible or not desired. This case, where only a certain subset of the eigenvalues for the closed loop is assigned, is referred to as *partial eigenvalue placement* [8], [9]. To calculate the state feedback for a stabilizable system one computes a Kalman decomposition into a controllable and an uncontrollable subsystem, respectively, and carries out eigenvalue assignment for the controllable subsystem. This approach to partial eigenvalue placement can be interpreted as a full eigenvalue assignment in the controllable subspace. In general, partial eigenvalue placement usually relies on projection or subspace methods, see [8], [9].

Controller design procedures for state space systems (1), which are available in numerical software packages such as MATLAB, GNU OCTAVE and SCILAB are based on linear algebra [10], [11]. Although we consider a linear controller for a linear system, multivariable controller design is intrinsically a multilinear problem [3], [12], which may lead to a linear problem only under special conditions [5], [13]. This type of problems can be solved by means of Gröbner bases [5], [14] using the framework of algebraic geometry [15]–[17].

In general, a small feedback gain is considered advantageous for robust controller design [11]. Several authors developed numerical methods to minimize the Frobenius norm of the feedback gain matrix [18]–[21]. The spectral norm is often used in a similar context. We discuss the minimization of both norms using polynomial ideals.

For a digital implementation of the control law using integers it would be advantageous if all entries of the gain matrix have values in the same order of magnitude. This and related goals can be addressed by minimizing the maximum matrix norm as well as the maximum absolute row and column sum norms of the gain matrix. This optimization problem can be solved using quantifier elimination [22], [23].

In Section II we consider the mathematical formulation of our feedback design goals. Computation methods are discussed in Section III. Our approach will be demonstrated on several example systems in Section IV. In Section V we summarize

This work was supported by Deutsche Forschungsgemeinschaft (DFG) under the research grant RO2427/5-1.

**Cite as:** K. Röbenack and D. Gerbet, "Full and Partial Eigenvalue Placement for Minimum Norm Static Output Feedback Control", *Syst. Theor. Control Comput. J.*, vol. 2, no. 1, pp. 22–33, Jun. 2022.
**DOI:** 10.52846/stccj.2022.2.1.32

the results.

## II. Feedback Design Considerations

System (1) with the static output feedback (2) results in the closed-loop system

$$\dot{x} = (A - BKC)\,x \tag{3}$$

with the characteristic polynomial

$$\begin{aligned} \mathrm{CP}(s) &= \det(sI - (A - BKC)) \\ &= a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n. \end{aligned} \tag{4}$$

In this section we will discuss possible design goals.

### A. Full and Partial Eigenvalue Placement

Some design procedures require certain constellations of eigenvalues. For example, the complete modal synthesis requires pairwise distinct eigenvalues [24], [25]. In this section, conditions for the assignment of different eigenvalue constellations are derived.

*Distinct Eigenvalues:* Assume we want to assign $l \leq n$ eigenvalues $s_1, \ldots, s_l$ to the closed-loop system (3). We temporarily assume that the desired eigenvalues are distinct and real. A number $s_i$ is an eigenvalue of the closed-loop system (3) if and only if the characteristic polynomial (4) contains the linear factor $(s - s_i)$. This factorization can be stated by the remainders of polynomial divisions

$$\mathrm{CP}(s) \bmod (s - s_i) \overset{!}{=} 0 \quad \text{for} \quad i = 1, \ldots, l. \tag{5}$$

In (5), the polynomial division to compute the remainder is carried out for each linear factor separately. The associated remainders can also be calculated successively

$$\begin{array}{rcl} \mathrm{CP}(s) &=& (s - s_1)\cdot Q_1(s) \;+\; R_1 \\ Q_1(s) &=& (s - s_2)\cdot Q_2(s) \;+\; R_2 \\ &\vdots& \\ Q_{l-1}(s) &=& (s - s_l)\cdot Q_l(s) \;+\; R_l \end{array} \tag{6}$$

resulting in the condition

$$R_i \overset{!}{=} 0 \quad \text{for} \quad i = 1, \ldots, l. \tag{7}$$

Note that for the polynomial divisions in (5) and (6) the divisor polynomial $(s - s_i)$ is a linear factor in $s$, i.e., a polynomial of degree one. The residual polynomial is of lower degree and therefore constant w.r.t. $s$, i.e., the remainders in (5) and (6) depend only on the entries $k_{ij}$ of the feedback gain matrix $K$. The formulas in (5) and (7) may be different, but for district zeros $s_1, \ldots, s_l$ these polynomials generate the same ideal. Hence, they have the same zero set, i.e., the same algebraic variety.

*Multiple Eigenvalues:* Assume that the eigenvalue $s_i \in \mathbb{Q}$ has the algebraic multiplicity of $\nu \geq 2$. In this case, the characteristic polynomial (4) should contain the factor $(s - s_i)^\nu$. Then, the successive division scheme according to (6) can be carried out as above, where the divisor $(s - s_i)$ is used exactly $\nu$ times. Alternatively, we could replace the division by linear factors in (5) by

$$\tilde{R}_i(s) := \mathrm{CP}(s) \bmod (s - s_i)^\nu \overset{!}{\equiv} 0, \tag{8}$$

where the resulting remainder $\tilde{R}_i(s)$ is polynomial of degree less than $\nu$ in $s$. The remainder $\tilde{R}_i(s)$ must be the zero polynomial, i.e., all coefficients of $\tilde{R}_i(s)$ w.r.t. $s$ are set to zero.

*Complex Conjugate Pair:* Now, we consider a conjugate complex pair $s_i = \sigma + j\omega$ and $s_{i+1} = \sigma - j\omega$. Instead of using two complex linear factors as in (5) we now use a real quadratic factor as divisor

$$\begin{aligned} \tilde{R}_i(s) &:= \mathrm{CP}(s) \bmod (s - s_i)(s - s_{i+1}) \\ &= \mathrm{CP}(s) \bmod (s - \sigma - j\omega)(s - \sigma + j\omega) \\ &= \mathrm{CP}(s) \bmod (s^2 - 2\sigma s + \sigma^2 + \omega^2) \overset{!}{\equiv} 0. \end{aligned} \tag{9}$$

The resulting remainder $\tilde{R}_i(s)$ is a polynomial of degree one w.r.t. the variable $s$. Setting both coefficients to zero yields two algebraic equations in the variables $k_{ij}$. This approach can easily be extended to multiple complex conjugate pairs.

### B. Stability

If the $l < n$ eigenvalues $s_1, \ldots, s_l$ are assigned as roots of the characteristic polynomial (4), the polynomials $(s - s_i)$ for $i = 1, \ldots, l$ are factors of (4). Therefore, the polynomial division

$$\begin{aligned} Q_l(s) &= \mathrm{CP}(s) \operatorname{div} (s - s_1)\cdots(s - s_l) \\ &= q_0 + q_1 s + \cdots + q_{n-l-1}s^{n-l-1} + s^{n-l} \end{aligned} \tag{10}$$

with the quotient polynomial $Q_l(s)$ is carried without remainder. Note that $Q_l(s)$ occurs also in the last step of the division scheme (6). The polynomial $Q_l(s)$ describes the dynamics not covered by the partial eigenvalue placement mentioned above. We want to ensure that $Q_l(s)$ is a Hurwitz polynomial, i.e., that all roots are in the open left half complex plane. Conditions on the coefficients for this stability property can be derived using the Routh or Hurwitz test

$$\begin{array}{ll} n - l = 2: & q_0 > 0 \wedge q_1 > 0, \\ n - l = 3: & q_0 > 0 \wedge q_1 > 0 \wedge q_1 q_2 - q_0 > 0, \\ n - l = 4: & q_0 > 0 \wedge q_1 > 0 \wedge q_2 > 0 \\ & \wedge\, q_1 q_2 q_3 - q_1^2 - q_0 q_3^2 > 0, \\ \vdots & \end{array} \tag{11}$$

see [26]–[28]. Similar conditions can be derived for the stability with purely real eigenvalues, i.e., with roots on the open left half real axis, see [28]–[31].

### C. Matrix Norms

The minimization (or at least a reduction) of the norm of the feedback gain matrix is a general goal in robust control [18]–[21]. In general, a ‘softer‘ control law is considered advantageous in robust control [32].

*1) Frobenius Norm:* In the authors opinion, the *Frobenius norm*

$$\|K\|_{\mathrm{F}} = \sqrt{\sum_{ij} k_{ij}^2} \tag{12}$$

is the most commonly used matrix norm in control theory for such type of design objectives [11], [21]. For the case of state feedback (i.e., $C = I$), the minimization w.r.t. (12) can be formulated via Sylvester's equation [18]–[20]. Also, general numerical optimization methods can be used. For later considerations we would like to point out that the value $g = \|K\|_{\mathrm{F}}$ can equivalently be characterized by

$$g \geq 0 \wedge g^2 = \sum_{ij} k_{ij}^2. \tag{13}$$

The Frobenius norm is often used in numerical linear algebra. The norm is submultiplicative due to the Cauchy-Schwarz inequality.

*2) Spectral Norm:* The Frobenius norm (12) can be seen as a generalization of the *Euclidean norm*

$$\|y\|_2 = \sqrt{\sum_i y_i^2} \tag{14}$$

of a vector $y \in \mathbb{R}^r$. This vector norm induces the matrix norm

$$\|K\|_2 = \max_{\|y\|_2=1} \|Ky\|_2, \tag{15}$$

which is called *spectral norm*. The norm (15) can also be characterized as the largest value $g$ with

$$g \geq 0 \wedge \sum_i y_i^2 = 1 \wedge g^2 = \sum_i \left( \sum_j k_{ij} y_j \right)^2. \tag{16}$$

The spectral norm (15) can also be characterized by means of singular values. Consider the characteristic equation

$$\det(sI - K^T K) = 0 \tag{17}$$

of a symmetric eigenvalue problem. While the matrix $K \in \mathbb{R}^{m \times r}$ may not be square, the matrix product $K^T K$ is square, symmetric and positive semidefinite. Hence, all roots $s_1 \geq s_2 \geq \cdots$ of (17) are real and non-negative. The singular values $\sigma_1 \geq \sigma_2 \geq \cdots$ of $K$ are the square roots of these eigenvalues, i.e., $\sigma_i = \sqrt{s_i}$ for $i = 1, \ldots, \min\{m, r\}$. Let $\bar{\sigma}(K) := \sigma_1$ denote the largest singular value of the matrix $K$. Then, the spectral norm (15) can be written as

$$\|K\|_2 = \bar{\sigma}(K). \tag{18}$$

The norms (12) and (15) are related by

$$\|K\|_2 \leq \|K\|_{\mathrm{F}}.$$

*3) Maximum Norm:* For a fast implementation of the control law (2), all measured and control variables are normalized to a certain range. This range is mainly determined by the integer range of the analog-to-digital and digital-to-analog converters. For the most uniform exploitation of the available measuring and control range, we would like to suggest the use of the *maximum norm*

$$\|K\|_{\max} = \max_{ij} |k_{ij}|. \tag{19}$$

The norm (19) can also be characterized as the smallest value $g$ fulfilling

$$\bigwedge_{i,j} (-g \leq k_{ij} \leq g). \tag{20}$$

Note that this matrix norm is not submultiplicative [33, p. 56].

*4) Maximum Absolute Row and Column Sum Norms:* The vector norms

$$\|y\|_\infty = \max_i |y_i| \quad \text{and} \quad \|y\|_1 = \sum_i |y_i|$$

induce the *absolute row sum norm*

$$\|K\|_\infty = \max_{\|y\|_\infty=1} \|Ky\|_\infty = \max_i \sum_j |k_{ij}| \tag{21}$$

and the *absolute column sum norm*

$$\|K\|_1 = \max_{\|y\|_1=1} \|Ky\|_1 = \max_j \sum_i |k_{ij}|, \tag{22}$$

respectively. Introducing numbers $g_{ij}$ with

$$-g_{ij} \leq k_{ij} \leq g_{ij}, \tag{23}$$

the norm (21) can be characterized as the minimum value $g$ with

$$\bigwedge_i \sum_j g_{ij} \leq g. \tag{24}$$

Similarly, the norm (22) can be characterized as the minimum value $g$ with

$$\bigwedge_j \sum_i g_{ij} \leq g. \tag{25}$$

The norms (21) and (22) are related by

$$\|K\|_\infty = \|K^T\|_1 \quad \text{and} \quad \|K\|_1 = \|K^T\|_\infty.$$

*Remark* 1. With the assignment of $n$ eigenvalues with negative real parts, the additional minimization of the norm of the gain matrix will always lead to a stable closed loop system as well. For a partial placement of $l < n$ eigenvalues with negative real parts, the situation may be different. If the solution with minimum matrix norm is an interior point in the stability region of the parameter space of the entries of the gain matrix $K$ according to (11), then the controller design problem is solved. Otherwise, we want to avoid solutions on the border of the stability region. In this case, we would replace the strict stability condition (11) by a robust stability condition, e.g. ensuring a certain margin of the eigenvalues to the imaginary axis, see [34], [35].

### D. System Norm

To study the dynamics of the closed-loop system we augment the feedback (2) by an additional reference input signal $w$ resulting in the control law

$$u = -Ky + w. \tag{26}$$

The transfer function of the closed-loop system, i.e., from the reference signal $w$ to the output signal $y$, is a $r \times m$-matrix

$$T(s) = C\left(sI - (A - BKC)\right)^{-1} B. \tag{27}$$

We assume that the gain matrix $K$ was chosen such that the closed-loop system is bounded-input bounded-output (BIBO) stable. Then, the transfer function (27) belongs to the *Hardy space* $\mathcal{H}_\infty$, see [36]. This vector space is equipped with the norm

$$\|T\|_\infty = \sup_\omega \bar{\sigma}(T(j\omega)). \tag{28}$$

In this equation, the frequency response $T(j\omega)$ is a complex matrix, from which the matrix spectral norm $\|T(j\omega)\|_2$ is computed according to (18). Then, the supremum is computed over all radian frequencies $\omega \in \mathbb{R}$.

The frequency response $T(j\omega)$ can be interpreted as a complex matrix-valued frequency dependent gain factor for the system under vector-valued harmonic excitation. Roughly speaking, the system norm (28) is the largest possible gain factor of the system.

Note that the norms (21) and (28) should not be confused, even though the same notation is used. The norm (21) is a matrix norm in a finite dimensional real or complex vector space, whereas (28) is a norm in an infinite dimensional function space.

## III. Computation Methods

The Frobenius and the spectral norm are minimized by a constrained optimization with multivariate polynomials. To minimize the maximum norm as well as absolute row and column sum norm, quantifier elimination is used. Both techniques belong to the field of computational algebra [15], [16].

### A. Constrained Optimization with Polynomials

The minimization of the Frobenius norm (12) characterized by (13) under equality constraints such as (5) or (7) can be formulated as a constrained optimization problem with the *Lagrangian function*

$$L(K, \lambda) = \sum_{ij} k_{ij}^2 + \sum_{i=1}^{l} \lambda_i R_i(K), \tag{29}$$

where $\lambda$ denotes the vector of the *Lagrangian multipliers* [37]. Note that the conditions (8) to (9) can be treated similarly. With the variables $z = (k_{11}, \ldots, k_{mr}, \lambda_1, \ldots, \lambda_l)$, the first order necessary conditions for constrained optimization are given by

$$\frac{\partial}{\partial z_i} L(K, \lambda) \stackrel{!}{=} 0 \quad \text{for} \quad i = 1, 2, \ldots, N \tag{30}$$

with $N = mr + l$.

The Lagrangian (29) is a multivariate polynomial in the ring $\mathbb{Q}[z]$. Therefore, the partial derivatives occurring in (30) are also polynomials, which generate an *ideal*

$$\mathcal{I} = \left\langle \frac{\partial}{\partial z_1} L(K, \lambda), \ldots, \frac{\partial}{\partial z_N} L(K, \lambda) \right\rangle \subset \mathbb{Q}[z], \tag{31}$$

see [15]. The set of all (possibly complex) solutions of (30) is the *variety* $\mathcal{V}(\mathcal{I}) \subset \mathbb{C}^N$ of the ideal $\mathcal{I}$, and the *real variety* $\mathcal{V}(\mathcal{I}) \cap \mathbb{R}^N$ contains all real solutions. To solve the set of polynomial equations (30) we first compute a Gröbner basis of the ideal $\mathcal{I}$ with respect to lexicographic order. From this Gröbner basis we obtain the elimination ideals, where the $j$-th elimination ideal is defined by

$$\mathcal{I}_j = \mathcal{I} \cap \mathbb{Q}[z_{j+1}, \ldots, z_N]. \tag{32}$$

Using these elimination ideals we can compute the real solutions numerically. The required calculation can be carried out with the open source computer algebra packages MAXIMA [38] as well as SAGEMATH [39]. From a finite number of solutions in the variables $z$ one can easily select the values $k_{11}, \ldots, k_{mr}$ of the gain matrix $K$ such that the norm (12) is the smallest.

For an optimization w.r.t. the spectral norm (15) we define the Lagrangian

$$L(K, \lambda, y) = \sum_i \left( \sum_j k_{ij} y_j \right)^2 + \lambda_0 \left( \sum_i y_i^2 - 1 \right) + \sum_{i=1}^{l} \lambda_i R_i \tag{33}$$

with the addition Lagrangian multiplier $\lambda_0$ to take the conditions (16) into account. The variables of this constrained optimization problem are collected in the vector $z = (k_{11}, \ldots, k_{mr}, \lambda_0, \lambda_1, \ldots, \lambda_l, y_1, \ldots, y_r)$. Then, we compute the first order derivatives (30) with $N = (m+1)\,r + l + 1$ and proceed as above.

In order to obtain the spectral norm, we have to maximize the last term in (16) for all $y \in \mathbb{R}^r$ belonging to the unit sphere, i.e., $\|y\|_2 = 1$. On the other hand, we want to minimize this term over all admissible values of the entries $k_{ij}$ satisfying the constraints resulting from the eigenvalue placement. This results in a min-max problem.

Alternatively, we could use the eigenvalue problem (17) to characterize the spectral norm (15). This consideration leads to the Lagrangian

$$L(K, \lambda, s) = s + \lambda_0 \det(sI - K^T K) + \sum_{i=1}^{l} \lambda_i R_i \tag{34}$$

depending on the variables collected in the vector $z = (k_{11}, \ldots, k_{mr}, \lambda_0, \lambda_1, \ldots, \lambda_l, s)$. Otherwise, we proceed as above.

Since the method of Lagrange multipliers writes the gradient of the objective function as a linear combination of the gradients of the constraint functions (using the Lagrange multipliers $\lambda_i$), (30) may fail to hold at the optimal point, if said gradients of the constraint functions w.r.t. the independent variables $k_{ij}$ are linearly dependent. The preconditions for (30) to hold at the optimal point are called *constraint qualification*. There are different criterions on the constraints [40]–[42] with different generality. Among these we will use the weakest one, namely that the gradients of the constraints are linearly independent. The points that fail these conditions can be computed easily and in a similar manner:

The equations for that the gradient of $R_i$, $i = 1, \ldots, l$ w.r.t. the gains $K$ is a multiple of that of the other constraints $R_j$, $j \neq i$, can be written as

$$C_{ipq} = \frac{\partial R_i}{\partial k_{pq}} + \sum_{j=1, j\neq i}^{l} \mu_j \frac{\partial R_j}{\partial k_{pq}} = 0$$

for $p = 1, \ldots, m$ and $q = 1, \ldots, r$. These polynomials generate the ideal

$$\mathcal{J}_i = \langle R_1, \ldots, R_l, C_{i11}, \ldots, C_{imr} \rangle \tag{35}$$

in the polynomial ring containing additional auxiliary variables $\mu_j$, $j = 1, \ldots, l$, $j \neq i$. From these ideals we now can compute the real solutions of the $(l-1)$-th elimination ideal. The optimal value is sought not only in the real variety of (31), but also in those of the ideals (35). Note that in the case of (33) and (34) the additional constraints and the auxiliary variables must also be taken into account.

### B. Quantifier Elimination

The usage of quantifier elimination in control theory has been suggested first in [43]. A serious drawback was the huge computational effort [44]. The significant advances in computing technology and the improvement of the algorithms make it feasible for practical applications [27].

We consider multivariate polynomials over the rational field $\mathbb{Q}$. These polynomials can be represented exactly in computer algebra systems. In the model theoretic framework, a polynomial equation or inequality is called an *atomic formula*. The concatenation of atomic formulas by boolean operations (such as $\wedge$, $\vee$, $\neg$) yields a *quantifier-free formula*. Quantifier-free formulas describe *semi-algebraic sets*.

Consider a quantifier-free formula in $n+1$ variables. This formula describes a semi-algebraic set in $\mathbb{R}^{n+1}$. According to the *Tarski-Seidenberg-Theorem* [23], [45], the projection of this set down to $\mathbb{R}^n$ results again in a semi-algebraic set.

*Mini Example* 1. The polynomial equation

$$x^2 + 2xy + 3y^2 - 1 = 0 \tag{36}$$

of the ellipse shown in Fig. 1 describes a semi-algebraic set. The projection to the $x$-axis yields the polynomial inequality

$$-1.2247 \approx -\sqrt{\tfrac{3}{2}} \leq x \leq \sqrt{\tfrac{3}{2}} \approx 1.2247, \tag{37}$$

whereas the projection to the $y$-axis results in

$$-0.7071 \approx -\sqrt{\tfrac{1}{2}} \leq y \leq \sqrt{\tfrac{1}{2}} \approx 0.7071. \tag{38}$$

Both inequalities describe semi-algebraic sets.

*Remark* 2. Note that the projection property discussed above does not hold for algebraic sets, i.e, sets described by a finite number of polynomial equations (without inequalities). For example, while describes (36) is an algebraic set, the projections (37) and (38) cannot be formulated in terms of algebra sets.

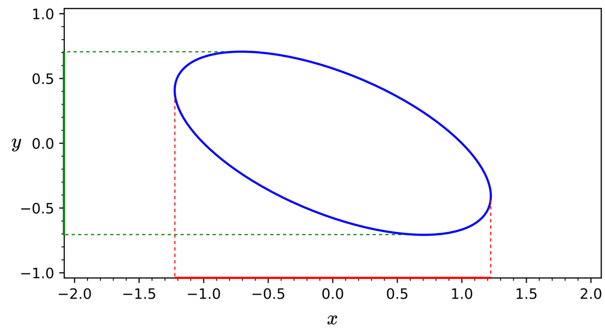


Fig. 1. Ellipse corresponding to (36)

Let $F(x, z)$ be a quantifier-free formula in the variables $x = (x_1, \ldots, x_n)$ and $z = (z_1, \ldots, z_m)$. A *prenex formula* is an expression of the form

$$(\mathsf{Q}_1 x_1)(\mathsf{Q}_2 x_2) \cdots (\mathsf{Q}_n x_n)\, F(x, z) \tag{39}$$

with *quantifiers* $\mathsf{Q}_i \in \{\forall, \exists, \exists!, \ldots\}$ for $i = 1, \ldots, n$. The variables $x_1, \ldots, x_n$ are called *quantified*, whereas $z_1, \ldots, z_m$ are *free variables*. As a consequence of the Tarski-Seidenberg-Theorem, each prenex formula (39) can equivalently be represented by a quantifier-free formula $H(z)$. By the transformation of (39) into the form $H(z)$, the quantifiers are eliminated along with the quantified variables $x$. This process is called *quantifier elimination* [22].

*Mini Example* 2. Consider the ellipse used in Mini Example 1. Applying quantifier elimination to the prenex formula

$$\exists y: \quad x^2 + 2xy + 3y^2 - 1 = 0$$

with the free variable $x$ and the quantified variable $y$ yields the equivalent quantifier-free formula $2x^2 - 3 \leq 0$, which corresponds to the interval (37). Similarly, quantifier elimination of

$$\exists x: \quad x^2 + 2xy + 3y^2 - 1 = 0$$

with the quantified variable $x$ and the free variable $y$ results in the equivalent quantifier-free formula $2y^2 - 1 \leq 0$ corresponding to (38).

During the last decades, several algorithms for quantifier elimination have been developed and improved. The most common algorithms are *cylindrical algebraic decomposition* (CAD) [46], *virtual substitution* (VS) [47], [48] and *real root classification* (RRC) [49], [50].

One of the first programs for solving non-trivial problems was QEPCAD, where a version of CAD was implemented [51]. The computations in this paper were carried out with the REDUCE package REDLOG [52], which combines a efficient implementation of VS with CAD as fallback solution [53]. Note that there are also quantifier elimination packages available for the commercial computer algebra system MAPLE, see [54], [55].

Quantifier elimination may lead to very large expressions, which can be simplified with the program SLFQ [56]. This program is based on QEPCAD B, see [57]. Note that SLFQ uses *extended Tarski formulas* to represent the simplified formulas.

Let $f(s) \in \mathbb{Z}[s]$ be a univariate polynomial in the variable $s$ with integers coefficients. The term

$$\mathrm{root}_j\, f(s) \tag{40}$$

with $j \in \mathbb{Z}$ denotes the $|j|$-th real root, where these roots are ordered from smallest to largest if $j > 0$ and largest to smallest if $j < 0$. This notation can be used for an exact representation of irrational roots.

*Mini Example* 3. To illustrate the extended Tarski notation we consider the polynomial $f(s) = s^3 - 8s^2 + 14s - 4$ with the roots $s_1 = 3 - \sqrt{7} \approx 0.354249$, $s_2 = 2$ and $s_3 = 3 + \sqrt{7} \approx 5.645751$. Using the notation (40), these real roots can be represented by

$$\begin{array}{lllll} s_1 &=& \mathrm{root}_{+1}\, f(s) &=& \mathrm{root}_{-3}\, f(s), \\ s_2 &=& \mathrm{root}_{+2}\, f(s) &=& \mathrm{root}_{-2}\, f(s), \\ s_3 &=& \mathrm{root}_{+3}\, f(s) &=& \mathrm{root}_{-1}\, f(s). \end{array}$$

## IV. EXAMPLES

In the Sections IV-A to IV-C we carry out full eigenvalue assignment, where we consider distinct, multiple and conjugate complex eigenvalues, respectively. In Section IV-D we perform a partial eigenvalue placement. The examples discussed were taken from the literature, where the eigenvalue placements were carried out without optimization. We already used these examples in the conference contribution [7], where the minimization of the Frobenius and maximum norm were discussed in more detail (see also [58]). Here, we also minimized the spectral norm as well as the maximum absolute row and column sum norms.

For the computations we employed a standard PC with Intel® Core™ i7-9700 CPU running at a clock frequency of 3 GHz with 32 GiB RAM under the Linux operating system Fedora 35 (x86-64). The polynomial ideals were computed with SAGEMATH 9.4. To carry out quantifier elimination we used REDUCE (CSL, rev 6261) with REDLOG.

### A. Example 1

We consider the system

$$A = \begin{pmatrix} -11.4 & -3.5 & 0 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 1.425 \\ 1 & -1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \tag{41}$$

discussed in [59, Example 1], where the eigenvalues are placed at $s_1 = -1$, $s_2 = -2$ $s_3 = -3$. The authors computed the gain matrix

$$K = \begin{pmatrix} 2.7827 & -3.4933 \\ 2.1837 & -3.0812 \end{pmatrix}. \tag{42}$$

The values of the different norm are given in line 0 of Table I.

For the eigenvalue placement we used the conditions (5) with the above mentioned $l = n = 3$ eigenvalues. First, we want to compute the gain matrices with the minimum Frobenius norm as well as minimum spectral norm using the Lagrangian functions (29) and (33), respectively. From (30) we obtain the polynomials generating the ideals (31). Using SAGEMATH [39], we computed the associated varieties. The real solutions with minimum norms are listed in the lines 1 and 2 of Table I. Both norm are reduced by a factor greater than three compared to the initial gain matrix (42).

Now, we want to minimize the norm (19) described by (20). As a first step, we want to determine the admissible range of the norm (19). Quantifying all entries $k_{ij}$ of the gain matrix yields the prenex formula

$$\exists k_{11}, k_{12}, k_{21}, k_{22} : \text{ Cond. (5)} \wedge \text{Cond. (20)}. \tag{43}$$

Using $g$ as a free variable results in 540 KiB code after quantifier elimination. This can be simplified with SLFQ to the condition

$$g \geq \mathrm{root}_{+1}\{9239795g^2 - 101272318g + 105598160\}, \tag{44}$$

where we used the notation introduced by (40) in Section III-B. The polynomial of degree two has two real solutions $g_1 \approx 1.166962$ and $g_2 \approx 9.793489$. The smallest real solution $g = g_1$ required in (44) can be described by the equivalent quantifier free formula

$$9239795g^2 - 101272318g + 105598160 = 0 \wedge g < 2. \tag{45}$$

We add this formula to the conditions (43) for eigenvalue placement and assign the existential quantifier to the variable $g$. Then, one of the entries $k_{ij}$ of the gain matrix is chosen as a free variable, i.e., we omit the existential quantifier for this variable. For example, quantifier elimination applied to the prenex formula

$$\exists g, k_{12}, k_{21}, k_{22} : \text{ Cond. (5)} \wedge \text{Cond. (20)} \wedge \text{Cond. (45)}.$$

yields a quantifier free formula in the variable $k_{11}$. Step by step, quantifier elimination results in the gain matrix shown in line 3 of Table I. The computed maximum norm is less than half of the value resulting from (42).

Finally, we want to compute the gain matrices minimizing the absolute row and column sum norms (21) and (22), respectively. Similar to the maximum norm, we want to determine the admissible range of these norms. For the absolute row sum norm (21), we have to take (23) and (24) into account. In particular, the bound (23) means

$$\begin{array}{lllll} -g_{11} & \leq & k_{11} & \leq & g_{11} \quad \wedge \\ -g_{12} & \leq & k_{12} & \leq & g_{12} \quad \wedge \\ -g_{21} & \leq & k_{21} & \leq & g_{21} \quad \wedge \\ -g_{22} & \leq & k_{22} & \leq & g_{22}, \end{array} \tag{46}$$

where (24) is stated as

$$g_{11} + g_{12} \leq g \wedge g_{21} + g_{22} \leq g. \tag{47}$$

TABLE I
ENTRIES OF THE GAIN MATRICES AND THE ASSOCIATED NORMS FOR EXAMPLE 1 (SECTION IV-A)

| $no$ | gain matrix $K$ | $\|K\|_{\mathrm{F}}$ | $\|K\|_2$ | $\|K\|_{\max}$ | $\|K\|_\infty$ | $\|K\|_1$ | $\|T\|_\infty$ |
|---|---|---|---|---|---|---|---|
| 0 | $\begin{pmatrix} 2.7827 & -3.4933 \\ 2.1837 & -3.0821 \end{pmatrix}$ | 5.849312 | 5.847063 | 3.4933 | 6.276 | 6.5754 | 2.554425 |
| 1 | $\begin{pmatrix} -0.247755 & -1.174350 \\ -1.157344 & -0.699223 \end{pmatrix}$ | **1.807993** | 1.661065 | 1.17435 | 1.856567 | 1.873573 | 2.039429 |
| 2 | $\begin{pmatrix} -0.217210 & -1.198257 \\ -1.126173 & -0.721447 \end{pmatrix}$ | 1.808799 | **1.659957** | 1.198257 | 1.84762 | 1.919704 | 2.019611 |
| 3 | $\begin{pmatrix} -0.257192 & -1.166962 \\ -1.166962 & -0.692366 \end{pmatrix}$ | 1.80807 | 1.661852 | **1.166962** | 1.859328 | 1.859328 | 2.045574 |
| 4 | $\begin{pmatrix} 0.200078 & -1.523553 \\ -0.694212 & -1.029419 \end{pmatrix}$ | 1.97557 | 1.854319 | 1.523553 | **1.723631** | 2.552972 | **1.775359** |
| 5 | $\begin{pmatrix} -0.380755 & -1.070096 \\ -1.292326 & -0.602986 \end{pmatrix}$ | 1.823123 | 1.690691 | 1.292326 | 1.895312 | **1.673082** | 2.126925 |

To compute the admissible values $g$ of the norm (21), we consider the prenex formula

$$\begin{array}{c} \exists g_{11}, g_{12}, g_{21}, g_{22}, k_{11}, k_{12}, k_{21}, k_{22}: \\ \text{Cond. (5)} \wedge \text{Cond. (46)} \wedge \text{Cond. (47)} \end{array} \tag{48}$$

with the quantified variables $g_{ij}$, $k_{ij}$ and the free variable $g$.

Alternatively, the bounds (46) and (47) can equivalently be written as

$$\begin{array}{rcccccc} -g & \leq & +k_{11}+k_{12} & \leq & g & \wedge \\ -g & \leq & -k_{11}+k_{12} & \leq & g & \wedge \\ -g & \leq & +k_{11}-k_{12} & \leq & g & \wedge \\ -g & \leq & -k_{11}-k_{12} & \leq & g & \wedge \\ -g & \leq & +k_{21}+k_{22} & \leq & g & \wedge \\ -g & \leq & -k_{21}+k_{22} & \leq & g & \wedge \\ -g & \leq & +k_{21}-k_{22} & \leq & g & \wedge \\ -g & \leq & -k_{21}-k_{22} & \leq & g, & \end{array} \tag{49}$$

where the auxiliary variables $g_{ij}$ are omitted. The corresponding prenex formula reads

$$\exists k_{11}, k_{12}, k_{21}, k_{22}: \text{ Cond. (5)} \wedge \text{Cond. (49)}. \tag{50}$$

Quantifier elimination of (48) or (50) results in 983 KiB code for the equivalent quantifier free formula, which can be simplified with SLFQ to the condition

$$g \geq \mathrm{root}_{+1}\{18829760q^2 - 119433867q + 149918508\}. \tag{51}$$

The polynomial occurring in (51) has two real roots $g_1 \approx 1.723631$ and $g_2 \approx 4.619194$, where (51) corresponds to $g \geq g_1$. To obtain the minimum allowed value of $g$ fulfilling the extended Tarski formula (51) we use the condition

$$18829760g^2 - 119433867g + 149918508 = 0 \wedge g < 2, \tag{52}$$

where the inequality $g < 2$ separates these the two roots. Then, we add (52) to the prenex formulas (48) or (50) and compute the elements $k_{ij}$ step by step. The gain matrix shown in line 4 of Table I.

For the absolute column sum norm (22), we have to take (23) and (25) into account. In our example, the bound (23) is already stated as (46), and (25) corresponds to

$$g_{11} + g_{21} \leq g \wedge g_{12} + g_{22} \leq g. \tag{53}$$

To calculate the admissible values $g$ of the norm (22), we consider the prenex formula

$$\begin{array}{c} \exists g_{11}, g_{12}, g_{21}, g_{22}, k_{11}, k_{12}, k_{21}, , k_{22}: \\ \text{Cond. (5)} \wedge \text{Cond. (46)} \wedge \text{Cond. (53).} \end{array} \tag{54}$$

Quantifier elimination resulted in 7.3 MiB code, which could not be simplified with SLFQ. However, using the equivalent formulation

$$\begin{array}{rcccccc} -g & \leq & +k_{11}+k_{21} & \leq & g & \wedge \\ -g & \leq & -k_{11}+k_{21} & \leq & g & \wedge \\ -g & \leq & +k_{11}-k_{21} & \leq & g & \wedge \\ -g & \leq & -k_{11}-k_{21} & \leq & g & \wedge \\ -g & \leq & +k_{12}+k_{22} & \leq & g & \wedge \\ -g & \leq & -k_{12}+k_{22} & \leq & g & \wedge \\ -g & \leq & +k_{12}-k_{22} & \leq & g & \wedge \\ -g & \leq & -k_{12}-k_{22} & \leq & g, & \end{array} \tag{55}$$

with the prenex formula

$$\exists k_{11}, k_{12}, k_{21}, k_{22}: \text{ Cond. (5)} \wedge \text{Cond. (55)} \tag{56}$$

yields 4.0 MiB code, which are simplified with SLFQ into

$$g \geq \mathrm{root}_{+1}\{351500g^2 - 1927565g + 2241054\}. \tag{57}$$

The polynomial has the real roots $g_1 \approx 1.673082$ and $g_2 \approx 3.810745$. Therefore, the minimum value of $g$ can be described by the polynomial bound

$$351500g^2 - 1927565g + 2241054 = 0 \wedge g < 2.$$

To compute the gain matrix we proceed as above. The result is shown in line 5 of Table I.

For each of the computed feedback gain matrices $K$, the transfer function (27) of the closed-loop system has the same poles. However, the input-output behaviour may differ. Fig. 2 shows the Bode magnitude plot of the transfer function (27) for different gain matrices $K$. For the initial gain matrix (42), the magnitude plot has a significant peak. The lowest gain over all frequencies is achieved with the $\|K\|_\infty$-optimal gain matrix.

The supremum over of the magnitude over all frequencies is the system norm (28). The results are listed in the last column of Table I. As for the gain matrices listed there, the largest system norm occurs using the initial gain matrix (42), whereas the $\|K\|_\infty$-optimal gain matrix results in the lowest system norm.

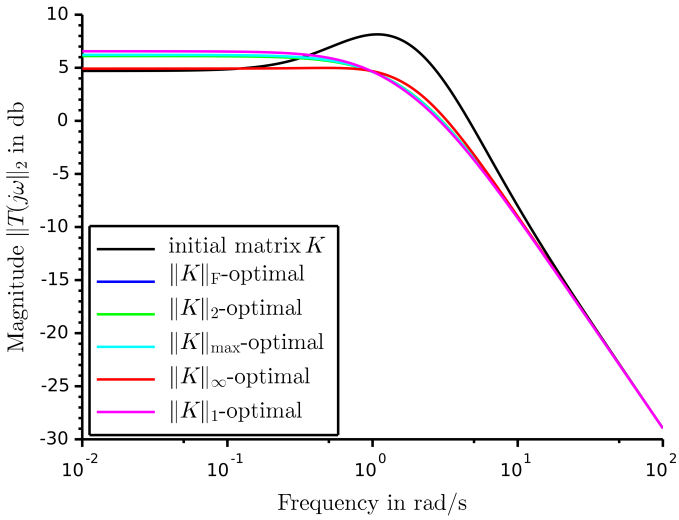


Fig. 2. Bode magnitude plot of the transfer function (27) for different gain matrices $K$ of Example 1 (Section IV-A)

## B. Example 2

Consider the system

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 19.62 & 0 & -8.86 \\ 0 & 0 & -100 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \tag{58}$$

taken from [59, Example 2]. As suggested in [59], we want to place the eigenvalues at $s_1 = s_2 = -3$ and $s_3 = -4$, which includes a double eigenvalue. In the paper mentioned, the authors computed the gain matrix

$$K = \begin{pmatrix} -44.7401 & -11.3932 \\ 0.5199 & -0.1689 \end{pmatrix}. \tag{59}$$

The values of the different norm are listed in Table II.

To derive conditions for the eigenvalue placement we carry out the polynomial division according to scheme (6), twice with the divisor $(s+3)$ and once with $(s+4)$. This results in constraints of the form (7). We computed the gain matrices with minimum Frobenius, spectral and maximum as described. The results are given in the lines 1 to 3 in Table II.

Next, we want to determine the admissible range for the absolute row sum norm (21), where we take the bounds (23) and (24) into account. These considerations result in the prenex formula

$$\begin{aligned} &\exists g_{11}, g_{12}, g_{21}, g_{22}, k_{11}, k_{12}, k_{21}, , k_{22} : \\ &\text{Cond. (7)} \wedge \text{Cond. (46)} \wedge \text{Cond. (47)}. \end{aligned} \tag{60}$$

Quantifier elimination with a subsequent simplification with SLFQ results in the condition

$$g > \mathrm{root}_{-1}\{228195g^2 - 19617029g + 420469392\}. \tag{61}$$

The polynomial of degree two has the real roots $g_1 \approx 40.757391$ and $g_2 \approx 45.208686$. Hence, this condition can equivalently be stated as

$$228195g^2 - 19617029g + 420469392 > 0 \wedge g > 45. \tag{62}$$

However, the set of admissible values $g$ described by the lower bound (61) has no minimum because it is an open interval. On the other hand, the eigenvalues depend continuously on the entries of the matrix. Therefore, we can approximate a minimum norm solution by the condition $g = g_2 + \varepsilon$ with a small number $\varepsilon > 0$. In practice, we round up the numerically determined value $g_2$. In our case, we use the condition $g := 45.208687$, which is consistent with (61) and (62). With this additional constraint, we determine the entries $k_{ij}$ of the matrix $K$ step by step. For each entry we now obtain a small interval, where we select one value. The results are given in the line 4 in Table II.

The computation of the admissible range for the absolute column sum norm (22) is carried out similarly. Quantifier elimination with a subsequent simplification yields

$$g > 45. \tag{63}$$

Again, the set (63) of admissible values for the norm has the infimum 45 but no minimum. Therefore, we add a small value $\varepsilon > 0$ to the infimum and specify the condition (63) to $g := 45.000001$, which is then used to compute the entries of the gain matrix $K$. The results are given in the line 5 in Table II.

The Bode magnitude plots of the transfer functions (27) for different gain matrices $K$ are shown in Fig. 3. The $\|K\|_{\max}$-optimal gain matrix yields the largest system magnitude and the $\|K\|_2$-optimal gain matrix the lowest. This result can also be seen in the last column of Table II, where the values of the corresponding system norm (28) are shown.

## C. Example 3

The system matrices of the third example are given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \ K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{pmatrix} \tag{64}$$

TABLE II
ENTRIES OF THE GAIN MATRICES AND THE ASSOCIATED NORMS FOR EXAMPLE 2 (SECTION IV-B)

| $no$ | gain matrix $K$ | $\|K\|_{\mathrm{F}}$ | $\|K\|_2$ | $\|K\|_{\max}$ | $\|K\|_\infty$ | $\|K\|_1$ | $\|T\|_\infty$ |
|---|---|---|---|---|---|---|---|
| 0 | $\begin{pmatrix} -44.7401 & -11.3932 \\ 0.5199 & -0.1689 \end{pmatrix}$ | 46.171207 | 46.170284 | 44.7401 | 56.1333 | 45.26 | 13.113644 |
| 1 | $\begin{pmatrix} -45.080824 & -2.273451 \\ -0.161647 & -0.884756 \end{pmatrix}$ | **45.147072** | 45.138583 | 45.080824 | 47.354274 | 45.242471 | 6.024498 |
| 2 | $\begin{pmatrix} -45.078102 & -2.326444 \\ -0.156204 & -0.888482 \end{pmatrix}$ | 45.147109 | **45.138546** | 45.078102 | 47.404546 | 45.234306 | **6.022228** |
| 3 | $\begin{pmatrix} -41.591684 & 5.301424 \\ 6.816632 & -41.591684 \end{pmatrix}$ | 59.450046 | 47.657611 | **41.591684** | 48.408316 | 48.408316 | 53.497508 |
| 4 | $\begin{pmatrix} -45.208686 & 0 \\ -0.417372 & -0.60688 \end{pmatrix}$ | 45.214686 | 45.210613 | 45.208686 | **45.208686** | 45.626058 | 6.83164 |
| 5 | $\begin{pmatrix} -45 & -3.946595 \\ 0 & -0.948079 \end{pmatrix}$ | 45.182679 | 45.172807 | 45 | 48.946595 | **45** | 6.329109 |

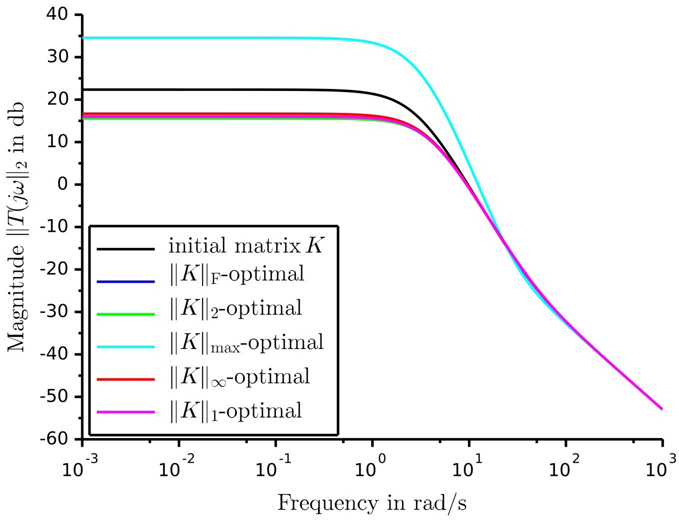


Fig. 3. Bode magnitude plot of the transfer function (27) for different gain matrices $K$ of Example 2 (Section IV-B)

see [60, Example 3]. The desired eigenvalues of the closed-loop system should be located at $-3, -4, -5, -2 \pm 2j$, which includes a complex conjugate pair. In [60], the authors numerically computed the feedback gain matrix

$$K = \begin{pmatrix} 5 & 11 \\ 24.3999 & 98.0002 \\ 137.001 & 370 \end{pmatrix} \quad (65)$$

where the values of the different matrix norms are given in Table III.

For the placement of the $n = 5$ eigenvalues, we have 6 entries of the gain matrix $K$. This degree of freedom can be used to optimize the gain matrix. The Frobenius norm and the maximum optimal gain matrices computed in [7], [61], respectively, are given in line 1 and 3 of Table III.

To minimize the spectral norm, we used the Lagrangian (29) and computed the ideal (31). The associated variety is zero-dimensional and consists of 20 points, of which 8 points correspond to a real gain matrix. From these points we selected the optimal solution given in line 2 of Table III.

We were not able to compute a symbolic expression for the admissible range for the absolute row sum norm (21). However, we were able to verify whether a certain rational value $g > 0$ is feasible. Using a bisection method, we were able to obtain the approximate value $g \approx 373.19722$. After fixing this value, we could compute the entries of the gain matrix $K$, see line 4 of Table III.

The computation of the admissible range for the absolute column sum norm (22) with REDLOG resulted in 248 MiB source code. SLFQ simplifies these expressions with 2255416 QEPCAD B calls to

$$\begin{aligned} g &\geq \mathrm{root}_{-1}\{133g^2 - 51480g - 1729800\} \\ &\gtrapprox \max\{-31.102227, 418.169896\} \\ &\approx 418.169896. \end{aligned} \quad (66)$$

The minimum feasible values can be described by the quantifier free formula

$$133g^2 - 51480g - 1729800 = 0 \wedge g > 300.$$

The computed gain matrix is shown in line 5 of Table III

Similar as for the other examples, we computed the Bode magnitude plots of the transfer functions (27) for different gain matrices $K$. The results are shown in Fig. 4. The optimal gain w.r.t. the Frobenius norm yields almost the same result as in case of for the minimum spectral norm achieving a small system norm (28). The $\|K\|_1$-optimal gain matrix yields the largest system magnitude. These results are shown in the last column of Table III, where the values of the corresponding system norm (28) are listed.

TABLE III

ENTRIES OF THE GAIN MATRICES AND THE ASSOCIATED NORMS FOR EXAMPLE 3 (SECTION IV-C)

| $no$ | gain matrix $K$ | $\|K\|_F$ | $\|K\|_2$ | $\|K\|_{\max}$ | $\|K\|_\infty$ | $\|K\|_1$ | $\|T\|_\infty$ |
|---|---|---|---|---|---|---|---|
| 0 | $\begin{pmatrix} 5 & 11 \\ 24.3999 & 98.0002 \\ 137.001 & 370 \end{pmatrix}$ | 407.44898 | 407.30463 | 370 | 507.001 | 479.0002 | 0.822321 |
| 1 | $\begin{pmatrix} 1.895956 & 14.104044 \\ -3.907739 & 101.10404 \\ 17.352556 & 373.10404 \end{pmatrix}$ | **387.23062** | 387.13934 | 373.10404 | 390.4566 | 488.31213 | **0.778624** |
| 2 | $\begin{pmatrix} 1.902063 & 14.097937 \\ -3.863767 & 101.09794 \\ 17.533751 & 373.09794 \end{pmatrix}$ | 387.23067 | **387.13929** | 373.09794 | 390.63169 | 488.29381 | 0.778629 |
| 3 | $\begin{pmatrix} 193.76136 & -177.76136 \\ 100.80929 & -90.761363 \\ -193.76136 & 181.23864 \end{pmatrix}$ | 397.40863 | 397.40039 | **193.76136** | 375 | 488.33202 | 2.079309 |
| 4 | $\begin{pmatrix} 194.59861 & -178.59861 \\ 101.67428 & -91.598606 \\ -192.79582 & 180.40139 \end{pmatrix}$ | 397.75487 | 397.74668 | 194.59861 | **373.19721** | 489.06871 | 2.089456 |
| 5 | $\begin{pmatrix} 99.579831 & -83.579831 \\ 0. & 3.420169 \\ -318.59006 & 275.42017 \end{pmatrix}$ | 440.75994 | 440.74861 | 318.59006 | 594.01023 | **418.1699** | 0.963403 |

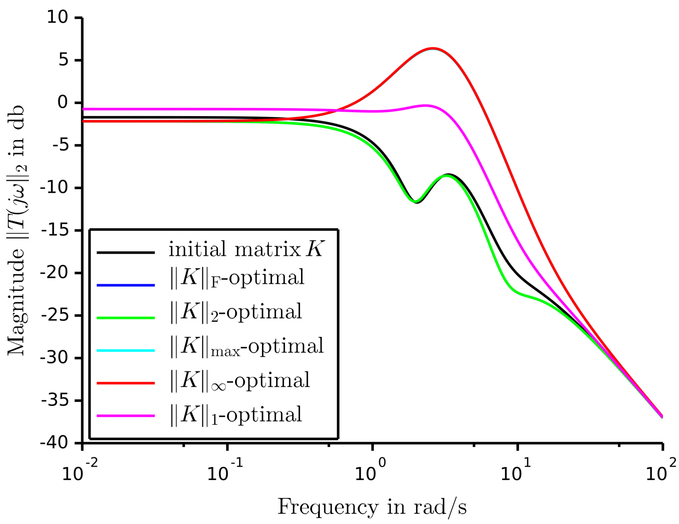


Fig. 4. Bode magnitude plot of the transfer function (27) for different gain matrices $K$ of Example 3 (Section IV-C)

## D. *Example 4*

We consider the system

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{67}$$
$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}.$$

taken from [5, Example 4.2] with $n = 4$. The open-loop system having the eigenvalues $\pm 1, \pm 2$ is not stable. It was shown [5] that system (67) is stabilizable by static output feedback, but a feedback matrix was not presented. The stabilization of this system is also discussed in [28].

Here, we want to perform a partial pole placement. For the closed-loop system, $l = 2$ of the four eigenvalues are placed at $s_1 = -3$ and $s_2 = -4$, where (5) leads to the following equality constraints:

$$\begin{aligned} k_{11}k_{22} - k_{12}k_{21} + 5k_{21} + 4k_{12} + 39k_{11} - 20 = 0, \\ k_{11}k_{22} - k_{12}k_{21} + 6k_{21} + 5k_{12} + 36k_{11} - 30 = 0. \end{aligned} \tag{68}$$

For this partial eigenvalue condition, we already computed stabilizing gain matrices with minimum Frobenius norm as well as minimum maximum norm in [7]. The results are shown in Table IV. The two computed gain matrices differ only slightly.

Next, we wanted to compute an optimal gain matrix w.r.t. the spectral norm. The partial eigenvalue placement conditions (68) are incorporated in the Lagrangian function (33) using Lagrangian multipliers. Unfortunately, the optimal solution is not among the Karush-Kuhn-Tucker points. The same holds for the Lagrangian function (34). In the latter case the gradient of the characteristic polynomial (17) vanishes w.r.t. the gains $k_{ij}$ and the auxiliary variable $s$ evaluated at the gain matrix given in line 2 of Table IV. Thus, this gradient is clearly linear dependent and the optimal solution is found in the real variety of the ideal (35) corresponding to said constraint (with $\mu = 0$). The constraint is not regular at this point due to the multiple singular value of the gain matrix. It is noted that the approach using the unit vector $y$ in (15) leads to a positive-dimensional variety in such a case, since a continuum of unit vectors $y$ leads to the same norm $\|Ky\|_2$.

The computation of the gain matrix with optimal maximum absolute row and column sum norms resulted in the same

TABLE IV
ENTRIES OF THE GAIN MATRICES AND THE ASSOCIATED NORMS FOR EXAMPLE 4 (SECTION IV-D)

| $no$ | gain matrix $K$ | $\|K\|_{\mathrm{F}}$ | $\|K\|_2$ | $\|K\|_{\max}$ | $\|K\|_\infty$ | $\|K\|_1$ | $\|T\|_\infty$ |
|---|---|---|---|---|---|---|---|
| 1 | $\begin{pmatrix} -0.011708 & 4.652185 \\ 5.31269 & -0.004657 \end{pmatrix}$ | $\mathbf{7.061704}$ | 5.312792 | 5.31269 | 5.317347 | 5.324398 | 0.65377 |
| 2 | $\begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$ | 7.071068 | **5** | **5** | **5** | **5** | $\mathbf{0.644788}$ |

matrix as in case of the optimal maximum as well as spectral norm.

## V. SUMMARY

In this paper we used computation methods from algebraic geometry to compute minimum norm gain matrices for static output feedback control of linear time-invariant state space systems. In general, these methods require a very high computational effort. With modern computers and efficient algorithms, non-trivial problems can now be solved. The methods described have been successfully tested on several different example systems.

The paper addressed the controller design based on eigenvalue assignment. We considered the placement of distinct as well as multiple real eigenvalues or complex conjugate pairs. The structure of the eigenvectors is thereby not taken into account, as opposed to, for example, the complete modal synthesis [24], [25].

We optimized the feedback gain matrices w.r.t. five norms using two different computation methods. While a small controller gain is generally helpful for robust control, it is not obvious according to which criterion or norm an optimization is recommended. The procedure described in the article gives the option to compare several controllers, each with the same eigenvalue configuration.

In this paper we extended the results presented in a previous conference publication [7]. The approach used for the Frobenius norm was modified for an optimization regarding to the spectral norm. Similarly to the maximum norm, we achieved an optimization w.r.t. the maximum absolute column sum or the maximum absolute row sum norm resulting in the matrix norms. The optimizations w.r.t. the new matrix norms required a higher computational effort compared to the optimization w.r.t. the Frobenius norm and the maximum norm, respectively.

In addition to the various matrix norms of the controller gain, we also calculated the norm of the closed-loop transfer function. The examples considered do not reveal any direct relationship between the minimization of the norm of the gain matrix and the norm of the transfer function. Conversely, this means that the existing numerical methods of $\mathcal{H}_\infty$ controller design do not necessarily lead to a controller gain matrix with small entries.

## REFERENCES


[1] H. Kimura, "On pole assignment by output feedback," *International Journal of Control*, vol. 28, no. 1, pp. 11–22, 1978.

[2] X. Wang, "Pole placement by static output feedback," *Journal of Math. Systems, Estimation, and Control*, vol. 2, pp. 205–218, 1992.

[3] J. Rosenthal and J. Willems, "Open problems in the area of pole placement," in *Open Problems in Mathematical Systems and Control Theory*, ser. Communication and Control Engineering Series, V. D. Blondel, E. D. Sontag, M. Vidyasagar, and J. C. Willems, Eds. London: Springer–Verlag, 1999, pp. 181–191.

[4] M. Franke, "Eigenvalue assignment by static output feedback – on a new solvability condition and the computation of low gain feedback matrices," *International Journal of Control*, vol. 87, no. 1, pp. 64–75, 2014.

[5] A. G. Yannakoudakis, "The static output feedback from the invariant point of view," *IMA Journal of Mathematical Control and Information*, vol. 33, no. 3, pp. 639–668, 2016.

[6] U. Konigorski, "Parametric eigenvalue assignment by constant output feedback–a cascaded approach," *Automatisierungstechnik*, vol. 68, no. 10, pp. 817–825, 2020.

[7] K. Röbenack and D. Gerbet, "Minimum norm partial eigenvalue placement for static output feedback control," in *International Conference on System Theory, Control and Computing (ICSTCC 2021)*, Iaşi, Romania, Oct. 2021, pp. 212–219.

[8] Y. Saad, "Projection and deflation method for partial pole assignment in linear state feedback," *IEEE Transactions on Automatic Control*, vol. 33, no. 3, pp. 290–297, 1988.

[9] B. N. Datta and D. R. Sarkissian, "Partial eigenvalue assignment in linear systems: existence, uniqueness and numerical solution," in *Proceedings of the Mathematical Theory of Networks and Systems (MTNS), Notre Dame*, 2002.

[10] W. M. Wonham, *Linear Multivariable Control - A Geometric Approach*, ser. Lecture Notes in Economics ans Mathematical Science. Springer-Verlag, 1974, vol. 101.

[11] V. Mehrmann and H. Xu, "An analysis of the pole placement problem. II. The multi-input case," *Electronic Transactions on Numerical Analysis*, vol. 5, pp. 77–97, Sept. 1997, http://eudml.org/doc/119610.

[12] R. Brockett and C. Byrnes, "Multivariable Nyquist criteria, root loci, and pole placement: A geometric viewpoint," *IEEE Transactions on Automatic Control*, vol. 26, no. 1, pp. 271–284, 1981.

[13] V. A. Zaitsev, "Modal control of a linear differential equation with incomplete feedback," *Differential Equations*, vol. 39, no. 1, pp. 145–148, 2003.

[14] Z. Kukelova, P. Krsek, V. Smutny, and T. Pajdla, "Gröbner basis solutions to satellite trajectory control by pole placement," in *Proceedings of the Advanced Maui Optical and Space Surveillance Technologies Conference*, 2013, pp. 748–757.

[15] D. A. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, 4th ed. Switzerland: Springer International Publishing, 2015.

[16] T. Becker and V. Weispfenning, *Gröbner Bases*, 2nd ed. New York: Springer-Verlag, 1998.

[17] D. Plaumann, *Einführung in die Algebraische Geometrie*. Springer Spektrum, 2020.

[18] L. H. Keel, J. A. Fleming, and S. P. Bhattacharyya, "Minimum norm pole assignment via Sylvester's equation," *Contemp. Math*, vol. 47, pp. 265–272, 1985.

[19] A. Varga, "Robust and minimum norm pole assignment with periodic state feedback," *IEEE Transactions on Automatic Control*, vol. 45, no. 5, pp. 1017–1022, 2000.

[20] ——, "Robust pole assignment via Sylvester equation based state feedback parametrization," in *CACSD. Conference Proceedings. IEEE International Symposium on Computer-Aided Control System Design (Cat. No. 00TH8537)*, 2000, pp. 13–18.

[21] B. N. Datta, W.-W. Lin, and J.-N. Wang, "Robust and minimum norm partial pole assignment in vibrating structures with aerodynamics effects," in *42nd IEEE International Conference on Decision and Control (CDC)*, vol. 3, 2003, pp. 2358–2363.

[22] B. F. Caviness and J. R. Johnson, Eds., *Quantifier Elimination and Cylindical Algebraic Decomposition*. Wien: Springer, 1998.

[23] A. Tarski, "A decision method for elementary algebra and geometry," in *Quantifier elimination and cylindrical algebraic decomposition*, B. F. Caviness and J. R. Johnson, Eds. Wien: Springer, 1998, pp. 24–84.

[24] G. Roppenecker, "Complete modal synthesis of constant output feedback controllers (in german)," *Automatisierungstechnik*, vol. 37, no. 1-12, pp. 304–311, 1989.

[25] J. Deutscher and P. Hippe, "A frequency domain approach to complete modal synthesis," *IFAC Proceedings Volumes*, vol. 33, no. 13, pp. 55–60, 2000.

[26] L. Gröll, *Methodik zur Integration von Vorwissen in die Modellbildung*, ser. Schriftenreihe des Instituts für Angewandte Informatik / Automatisierungstechnik am Karlsruher Institut für Technologie. KIT Scientific Publishing, 2015, vol. 52.

[27] K. Röbenack and R. Voßwinkel, "Solution of control engineering problems by means of quantifier elimination (in german)," *Automatisierungstechnik*, vol. 67, no. 9, pp. 714–726, 2019.

[28] ——, "Eigenvalue placement by quantifier elimination — the static output feedback problem," *Acta Cybernetica*, vol. 24, no. 3, pp. 409–427, 2020. [Online]. Available: https://doi.org/10.14232/actacyb.24.3.2020.8

[29] H. Anai and S. Hara, "A robust control system design by a special quantifier elimination methods using a Sturm-Habicht sequence," *Proc. IMACS-ACA'99*, 1999.

[30] J. B. Lasserre, "Polynomials with all zeros real and in a prescribed interval," *Journal of Algebraic Combinatorics*, vol. 16, no. 3, pp. 231–237, 2002.

[31] K. Röbenack and R. Voßwinkel, "On real stable pole placement for structured systems using Sturm and Sturm-Habicht sequences," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 4546–4552, 2020.

[32] R. A. Freeman and P. V. Kokotović, "Design of 'softer' robust nonlinear control laws," *Automatica*, vol. 29, pp. 1425–1437, 1993.

[33] G. H. Golub and C. F. van Loan, *Matrix Computations*, 3rd ed. Baltimore: John Hopkins University Press, 1996.

[34] K. Reinschke, *Lineare Regelungs- und Steuerungstheorie*, 2nd ed. Berlin, Heidelberg: Springer-Verlag, 2014.

[35] R. Voßwinkel, L. Pyta, F. Schrödel, İ. Mutlu, D. Mihailescu-Stoica, and N. Bajcinca, "Performance boundary mapping for continuous and discrete time linear systems," *Automatica*, vol. 107, pp. 272–280, 2019.

[36] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Upper Saddle River, New Jersey: Prentice Hall, 1998.

[37] D. Liberzon, *Calculus of variations and optimal control theory: A Concise Introduction*. Princeton, New Jersey: Princeton Univerity Press, 2021.

[38] "Maxima, a Computer Algebra System," http://maxima.sourceforge.net.

[39] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 9.0)*, 2020, https://www.sagemath.org.

[40] O. L. Mangasarian and S. Fromivitz, "The fritz john necessary optimality conditions in the presence of equality and inequality constraints," *Journal of Mathematical Analysis and Applications*, vol. 17, pp. 37–47, Jan. 1967.

[41] R. Henrion, "On constraint qualifications," *Journal of Optimization Theory and Applications*, vol. 72, pp. 187–197, Jan. 1992.

[42] G. Giorgi, "A note on the guignard constraint qualification and the guignard regularity condition in vector optimization," *Applied Mathematics*, vol. 4, no. 4, pp. 734–740, 2013.

[43] B. Anderson, N. Bose, and E. Jury, "Output feedback stabilization and related problems-solution via decision methods," *IEEE Trans. on Automatic Control*, vol. 20, no. 1, pp. 53–66, Feb. 1975.

[44] J. H. Davenport and J. Heintz, "Real quantifier elimination is doubly exponential," *Journal of Symbolic Computation*, vol. 5, no. 1, pp. 29–35, 1988.

[45] A. Seidenberg, "A new decision method for elementary algebra," *Annals of Mathematics*, vol. 60, no. 2, pp. 365–374, 1954.

[46] G. E. Collins, "Quantifier elimination for real closed fields by cylindrical algebraic decompostion," in *Automata Theory and Formal Languages 2nd GI Conference Kaiserslautern, May 20–23, 1975*. Springer, 1975, pp. 134–183.

[47] V. Weispfenning, "The complexity of linear problems in fields," *Journal of Symbolic Computation*, vol. 5, no. 1-2, pp. 3–27, 1988.

[48] T. Sturm, "Thirty years of virtual substitution: Foundations, techniques, applications," in *Proc. of the 2018 ACM on International Symposium on Symbolic and Algebraic Computation*. ACM, 2018, pp. 11–16.

[49] L. González-Vega, T. Recio, H. Lombardi, and M.-F. Roy, "Sturm-Habicht sequences, determinants and real roots of univariate polynomials," in *Quantifier Elimination and Cylindrical Algebraic Decomposition*, B. F. Caviness and J. R. Johnson, Eds. Vienna: Springer, 1998, pp. 300–316.

[50] L. Yang, X. Hou, and Z. Zeng, "Complete discrimination system for polynomials," *Science in China Series E Technological Sciences*, vol. 39, no. 6, p. 628–646, 1996.

[51] G. E. Collins and H. Hong, "Partial cylindrical algebraic decomposition for quantifier elimination," *Journal of Symbolic Computation*, vol. 12, no. 3, pp. 299–328, 1991.

[52] A. Dolzmann and T. Sturm, "Redlog: Computer algebra meets computer logic," *ACM SIGSAM Bulletin*, vol. 31, no. 2, pp. 2–9, 1997.

[53] M. Košta, "New concepts for real quantifier elimination by virtual substitution," Dissertation, Universität des Saarlandes, Fakultät für Mathematik und Informatik, Saarbrücken, Germany, 2016, https://publikationen.sulb.uni-saarland.de/handle/20.500.11880/26735.

[54] H. Yanami and H. Anai, "Development of SyNRAC," in *Computational Science – ICCS 2006*, V. N. Alexandrov, G. D. van Albada, P. M. A. Sloot, and J. Dongarra, Eds. Berlin, Heidelberg: Springer, 2006, pp. 462–469.

[55] C. Chen and M. M. Maza, "Quantifier elimination by cylindrical algebraic decomposition based on regular chains," *Journal of Symbolic Computation*, vol. 75, pp. 74–93, 2016.

[56] C. W. Brown and C. Gross, "Efficient preprocessing methods for quantifier elimination," in *CASC*, ser. Lecture Notes in Computer Science, vol. 4194. Springer, 2006, pp. 89–100.

[57] C. W. Brown, "QEPCAD – Quantifier Elimination by Partial Cylindrical Algebraic Decomposition." [Online]. Available: https://www.usna.edu/CS/qepcadweb/B/QEPCAD.html

[58] "Minimum Norm Static Output Feedback." [Online]. Available: https://github.com/TUD-RST/minimum-norm-output-feedback

[59] S. Guney and A. Atasoy, "An approach to pole placement method with output feedback," in *UKACC Control Conference*, 2011. [Online]. Available: https://www.researchgate.net/publication/267424630_An_Approach_to_Pole_Placement_Method_with_Output_Feedback

[60] T. H. Lee, Q. G. Wang, and E. K. Koh, "An iterative algorithm for pole placement by output feedback," *IEEE Trans. on Automatic Control*, vol. 39, no. 3, pp. 565–568, Mar. 1994.

[61] K. Röbenack, R. Voßwinkel, and M. Franke, "On the eigenvalue placement by static output feedback via quantifier elimination," in *Mediterranean Conference on Control and Automation (MED'18)*, Zadar, Croatia, June 2018, pp. 133–138.