On functional differential equations connected to Huygens synchronization under propagation

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***Abstract*—The structure represented by one or several oscillators couple to a one-dimensional transmission environment (e.g. a vibrating string in the mechanical case or a lossless transmission line in the electrical case) turned to be attractive for the research in the field of complex structures and/or complex behavior. This is due to the fact that such a structure represents some generalization of various interconnection modes with lumped parameters for the oscillators.**

On the other hand the lossless and distortionless propagation along transmission lines has generated several research in electrical, thermal, hydro and control engineering leading to the association of some functional differential equations to the basic initial boundary value problems.

The present research is performed at the crossroad of the aforementioned directions. We shall associate to the starting models some functional differential equations - in most cases of neutral type - and make use of the general theorems for existence and stability of forced oscillations for functional differential equations. The challenges introduced by the analyzed problems for the general theory are emphasized, together with the implication of the results for various applications.

***Index Terms*—conservation laws, distortionless propagation, time delays**

I. INTRODUCTION

In order to discuss some basic aspects of the synchronization, we shall firstly refer to a classical, unfortunately less circulated book [1]. It is mentioned there that "based on dynamical systems' classification from the point of view of the structure of state space splitting, two kinds of dynamics behavior can be pointed out: stability/instability and synchronic/chaotic behavior. These kinds of behavior of a dynamical system might be connected with the structure of the interactions of its subsystems".

A. This pioneering idea will be met in several papers in the years that followed. For instance, it is stated from the beginning in [2] that "motivated by problems in physics, physiology and biology, there have been many studies in recent years devoted to the dynamics induced from the ordinary differential equations obtained by coupling large numbers of oscillators on periodic lattices". It is mentioned further that "in the modeling of the synchronization problem, the basic oscillators used at each point very often are taken to be described by an ordinary differential equation of the van der Pol type or Liénard type, that is the differential equation has a unique equilibrium point and a periodic orbit that attracts all other orbits". The same idea is to be found in [3]. In both these papers the structure of the interactions is linear e.g. diffusive i.e. each oscillator is coupled to its neighbors only or latticial i.e. according to some geometric structure.

A further generalization of the aforementioned structure is to consider oscillators in the sense of Lagrange i.e. any dynamical system described by ordinary differential equations (Lagrange used to call oscillation any transient i.e. non-steady state evolution of a system) regardless it displays or not a periodic orbit. These oscillators are connected by *complex interactions* among which one can mention the delayed couplings [4], [5] or distributed couplings - diffusive [6] or with propagation [7], [8].

B. According to systems' structure, we may distinguish two approaches of analysis. When the oscillators are genuine i.e. each of them has a periodic orbit, the approach is that of [9], [10]. The set of completely decoupled oscillators has an invariant torus T_0^N corresponding to the Cartesian product of the hyperbolically stable periodic orbits of the N individual oscillators; this torus is also hyperbolic and exponentially stable. If the couplings are *weak* i.e. a representative parameter is sufficiently small, the system will also have an invariant torus T_k^N which is hyperbolic and exponentially stable. The flow on this last torus is described by a system of ordinary differential equations involving N angles θ_j, $j = \overline{1,N}$. Synchronization is achieved when $\theta_j(t) - \theta_k(t) \to \text{const}$ for $t \to \infty$.

In the second case the e.g. periodic orbit are not inherent to individual oscillators but rather generated by the interconnections. Synchronization means in this case phase synchronization as above but also a unique frequency. If the oscillation is not harmonic one may think about a unique period for the entire system of coupled oscillators.

C. We have thus discussed the existence of an oscillatory solution synchronized for all oscillators e.g. existence of a compact global attractor. The stability of this attractor has been also mentioned. In fact, as emphasized by one of the classics of the stability theory - N.G. Četaev in [11], *only those steady states are physically observable and measurable which are at least stable if not more i.e. asymptotically or even exponentially stable*; this was called by Četaev the *"Stability*

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postulate".

II. Motivation and statement of the problem

The paper was stimulated by a report of A.S. Pikovsky at the *International Symposium on Topical Problems of Nonlinear Wave Physics NWP 2014*, mentioning the neutral functional differential equations occurring in a synchronization problem with time delay. This signified in fact that the delay was generated by propagation, i.e. by hyperbolic partial differential equations.The results on the subject are published in [7], [8] but in fact only [8] deals with two oscillators "hanging" on a string while in [7] only a single oscillator is present; for this reason [7] acts more as a "toy example" in our development.

Before starting the paper development it appears as necessary to discuss the mathematical approach of the paper and its basics. The classical method of d'Alembert applied to hyperbolic partial differential equations in two variables with complicated (non standard) boundary conditions was adapted in [12] as *integration along the characteristics* leading to some functional equations containing differential, delay differential or Volterra operators. The solutions of these equations are used to represent the solutions of the boundary value problems for hyperbolic partial differential equations. If the correspondence between the solutions of the two mathematical objects is one-to-one (injective), then all results obtained for one object can be projected back on the other one. Later and independently, Cooke and Krumme [13], [14] presented the same approach for simpler cases; the reader is also sent to [15] for the complete rigorous proof of the one-to-one correspondence in the aforementioned method of Cooke and Krumme.

We are now in position to state the problems discussed in this paper. There will be considered the models of [7], [8] - the two cases - viewed as boundary value problems for hyperbolic partial differential equations, together with initial conditions. It will appear throughout the paper development that the assumed *a priori* "incident wave coming from $-\infty$" is nothing more but the effect of the initial conditions (at $t = 0$) for the vibrating string. Also existence and exponential stability of the oscillation will follow as a consequence of a corresponding theorem of Oscillation Theory.

III. The simplest case of a single oscillator

Following [7] we shall consider the transverse oscillations of an infinite elastic spring where a localized un-damped nonlinear oscillator is "hanging" (it might be a nonlinear pendulum). Let $y(x,t)$ be the local transverse displacement of the string and $z(t)$ - the local oscillator coordinate. We shall have the model

$$\begin{aligned}&\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0 \ , \ c^2 = T/\rho \ ; \ t > 0 \ , \ -\infty < x < \infty \\ &m\ddot{z} + Q(z) = T\left(\frac{\partial y_r}{\partial x}(0,t) - \frac{\partial y_l}{\partial x}(0,t)\right) \\ &y_l(0,t) = y_r(0,t) = z(t)\end{aligned} \tag{1}$$

together with the initial conditions

$$\begin{aligned}&z(0) = z_0 \ , \ \dot{z}(0) = z_1 \ ; \ y(x,0) = y_0(x) \ , \\ &\frac{\partial y}{\partial t}(x,0) = y_1(x)\end{aligned} \tag{2}$$

Here ρ is the string density and T - the strain force; consequently $c = \sqrt{T/\rho}$ represents the speed of the wave propagation through the string; $y_r(x,t)$ and $y_l(x,t)$ are the displacements of the string for $x > 0$ and $x < 0$ respectively. Obviously the oscillator is "hanging" at $x = 0$. In the following we shall apply the methodology described in the Appendix.

A. Remark first that, starting from (1) and (2), we can define two initial boundary value problems in the LHS (left hand side) $\{(x,t)|x < 0, t > 0\}$ and RHS (right hand side) $\{(x,t)|x > 0, t > 0\}$ upper quadrants of the plane $\mathbb{R}\times\mathbb{R}$, namely

$$\begin{aligned}&\frac{\partial^2 y_l}{\partial t^2} - c^2\frac{\partial^2 y_l}{\partial x^2} = 0 \ ; \ t > 0 \ , \ x < 0 \\ &y_l(0,t) = z(t) \ ; \ y_l(x,0) = y_0(x) \ , \\ &\frac{\partial y_l}{\partial t}(x,0) = y_1(x) \ , \ x < 0\end{aligned} \tag{3}$$

and

$$\begin{aligned}&\frac{\partial^2 y_r}{\partial t^2} - c^2\frac{\partial^2 y_r}{\partial x^2} = 0 \ ; \ t > 0 \ , \ x > 0 \\ &y_r(0,t) = z(t) \ ; \ y_r(x,0) = y_0(x) \ , \\ &\frac{\partial y_r}{\partial t}(x,0) = y_1(x) \ , \ x > 0\end{aligned} \tag{4}$$

which can be tackled separately (such kind of problems are mentioned and are subject to some discussion in [16]). We shall apply to each of them the approach described in the Appendix.

Assume the solutions are smooth enough to introduce the variables

$$\frac{\partial y}{\partial t}(x,t) := v(x,t) \ , \ \frac{\partial y}{\partial x}(x,t) := w(x,t) \tag{5}$$

which are subject to the following PDE (partial differential equations) regardless the LHS or RHS quadrants

$$\frac{\partial v}{\partial t} = c^2\frac{\partial w}{\partial x} \ , \ \frac{\partial w}{\partial t} = \frac{\partial v}{\partial x} \tag{6}$$

a system in the symmetric Friedrichs form which can be diagonalized as follows

$$\begin{aligned}&u^+(x,t) = v(x,t) - cw(x,t) \ ; \\ &u^-(x,t) = v(x,t) + cw(x,t) \\ &v(x,t) = \frac{1}{2}(u^-(x,t) + u^+(x,t)) \ ; \\ &w(x,t) = \frac{1}{2c}(u^-(x,t) - u^+(x,t))\end{aligned} \tag{7}$$

Observe that $u^+(x,t)$ and $u^-(x,t)$ are the incident (progressive) and reflected (regressive) waves respectively. The initial conditions are deduced as follows

$$\begin{aligned} & u^-(x,0) = v(x,0) + cw(x,0) = y_1(x) + cy_0'(x) \ ; \\ & u^+(x,0) = y_1(x) - cy_0'(x) \end{aligned} \tag{8}$$

B. Consider first the problem in the RHS quadrant ($x > 0$)

$$\begin{aligned} & \frac{\partial u_r^+}{\partial t} + c\frac{\partial u_r^+}{\partial x} = 0 \ , \ \frac{\partial u_r^-}{\partial t} - c\frac{\partial u_r^-}{\partial x} = 0 \ ; \ x > 0 \ , \ t > 0 \\ & u_r^-(0,t) + u_r^+(0,t) = 2\dot{z}(t) \ , \ t > 0 \\ & u_r^+(x,0) = y_1(x) - cy_0'(x) \ , \\ & u_r^-(x,0) = y_1(x) + cy_0'(x) \ ; \ x > 0 \end{aligned} \tag{9}$$

Consider some point (x,t) crossed by two characteristics. The increasing characteristic $t^+(\xi;x,t) = t + (\xi - x)/c$ can be extended "to the left" where it can cross either the axis $x = 0$ or the axis $t = 0$: if $t > x/c$ it crosses $x = 0$ and if $t < x/c$ it crosses $t = 0$ at $\xi = x - ct > 0$. As follows from the methodology in the Appendix, the progressive wave should be integrated along the increasing characteristic lines - where the wave is constant. In this way it is obtained the representation of the progressive wave for $t > 0$, $x > 0$

$$u_r^+(x,t) = \begin{cases} y_1(x-ct) - cy_0'(x-ct) \ ; \ 0 < t < x/c \\ u_r^+(0, t - x/c) \ ; \ t > x/c \end{cases} \tag{10}$$

In order to make use of the boundary condition, the regressive wave is also required. This wave is constant along the decreasing characteristic lines. The decreasing characteristic line $t^-(\xi;x,t) = t - (\xi - x)/c$ can be extended "to the right" where it will cross the axis $t = 0$ at $\xi = x + ct > 0$. Therefore

$$u_r^-(x,t) = y_1(x+ct) + cy_0'(x+ct) \tag{11}$$

For $x = 0$ it follows that

$$y_1(ct) + cy_0'(ct) + u_r^+(0,t) = 2\dot{z}(t) \tag{12}$$

which allows the following inference for $t > x/c$

$$u_r^+(x,t) = 2\dot{z}(t - x/c) - y_1(ct - x) - cy_0'(ct - x) \tag{13}$$

Summarizing we obtained the following representation formulae for the solution of the RHS problem

$$\begin{aligned} & u_r^+(x,t) = \begin{cases} y_1(x-ct) - cy_0'(x-ct) \ ; \ 0 < t < x/c \\ 2\dot{z}(t-x/c) - y_1(ct-x) - cy_0'(ct-x) \ ; \\ t > x/c \end{cases} \\ & u_r^-(x,t) = y_1(x+ct) + cy_0'(x+ct) \end{aligned} \tag{14}$$

Since (13) was an inference we have to check that (14) is indeed a solution of (9); this is done by direct computation.

To end the analysis of this problem we write down the formulae for $v_r(x,t)$ and $w_r(x,t)$

$$\begin{aligned} & v_r(x,t) = \begin{cases} \frac{1}{2}[y_1(x+ct) + cy_0'(x+ct) + \\ +y_1(x-ct) + cy_0'(x-ct)] \ ; \ 0 < t < x/c \\ \frac{1}{2}[y_1(x+ct) + cy_0'(x+ct) + 2\dot{z}(t-x/c) - \\ -y_1(ct-x) - cy_0'(ct-x)] \ ; \ t > x/c \end{cases} \\ & w_r(x,t) = \begin{cases} \frac{1}{2c}[y_1(x+ct) + cy_0'(x+ct) - \\ -y_1(x-ct) - cy_0'(x-ct)] \ ; \ 0 < t < x/c \\ \frac{1}{2c}[y_1(x+ct) + cy_0'(x+ct) - 2\dot{z}(t-x/c) + \\ +y_1(ct-x) + cy_0'(ct-x)] \ ; \ t > x/c \end{cases} \end{aligned} \tag{15}$$

C. For $x < 0$ we shall have the boundary value problem

$$\begin{aligned} & \frac{\partial u_l^+}{\partial t} + c\frac{\partial u_l^+}{\partial x} = 0 \ , \ \frac{\partial u_l^-}{\partial t} - c\frac{\partial u_l^-}{\partial x} = 0 \ ; \ x < 0 \ , \ t > 0 \\ & u_l^-(0,t) + u_l^+(0,t) = 2\dot{z}(t) \ , \ t > 0 \\ & u_l^+(x,0) = y_1(x) - cy_0'(x) \ , \\ & u_l^-(x,0) = y_1(x) + cy_0'(x) \ ; \ x < 0 \end{aligned} \tag{16}$$

The approach is similar to the one for $x > 0$ but here the decreasing characteristic line can be extended "to the right" either up to the axis $x = 0$ (if $t > -x/c$) or up to the axis $t = 0$, the crossing point being in this last case $\xi = x + ct$ (if $0 < t < -x/c$); the increasing characteristic line, however, can be extended "to the left" up to the axis $t = 0$ which will be crossed at $\xi = x - ct < 0$. We obtain the representation of the regressive wave for $t > 0$, $x < 0$

$$u_l^-(x,t) = \begin{cases} y_1(x+ct) + cy_0'(x+ct) \ ; \ 0 < t < -x/c \\ u_l^-(0, t + x/c) \ ; \ t > -x/c \end{cases} \tag{17}$$

then of the progressive wave

$$u_l^+(x,t) = y_1(x-ct) - cy_0'(x-ct) \tag{18}$$

Using the definition of the boundary condition it follows that

$$\begin{aligned} & u_l^-(x,t) = \begin{cases} y_1(x+ct) + cy_0'(x+ct) \ ; \ 0 < t < -x/c \\ 2\dot{z}(t+x/c) - y_1(-ct-x) + cy_0'(-ct-x) \ ; \\ t > -x/c \end{cases} \\ & u_l^+(x,t) = y_1(x-ct) - cy_0'(x-ct) \end{aligned} \tag{19}$$

which verify (16) including the initial and the boundary conditions. Also

$$v_l(x,t) = \begin{cases} \frac{1}{2}[y_1(x+ct) + cy_0'(x+ct) + \\ +y_1(x-ct) - cy_0'(x-ct)] \ ; \ 0 < t < -x/c \\ \frac{1}{2}[y_1(x-ct) - cy_0'(x-ct) + 2\dot{z}(t-x/c) - \\ -y_1(-ct-x) - cy_0'(-ct-x)] \ ; \ t > -x/c \end{cases}$$

$$w_l(x,t) = \begin{cases} \frac{1}{2c}[y_1(x+ct) + cy_0'(x+ct) - \\ -y_1(x-ct) + cy_0'(x-ct)] \ ; \ 0 < t < -x/c \\ \frac{1}{2c}[-y_1(-x-ct) + cy_0'(-x-ct) + \\ 2\dot{z}(t+x/c) - y_1(x-ct) + cy_0'(x-ct)] \ ; \\ t > -x/c \end{cases} \quad (20)$$

D. It remains to consider here the equation of the "hanging" local oscillator

$$m\ddot{z} + Q(z) = T(w_r(0,t) - w_l(0,t)) \quad (21)$$

where we substitute the values of (19) and (20) to obtain

$$m\ddot{z} + (T/c)\dot{z} + Q(z) = (1/c)(y_1(ct) + y_1(-ct) + \\ +cy_0'(ct) - cy_0'(-ct)) \quad (22)$$

Let us remark that the coupling to the string introduces a local damping in the oscillator as well as a forcing term due to the initial displacement of the string. This corresponds to the description of [7]: introduction of the local damping is called in physical terms *radiation dissipation* while the forcing due to the initial displacement of the string appears as more realistic than the assumed "incident wave coming from $-\infty$".

It is now time to show that (22) describes a synchronization process. Denote

$$f(t) := (1/c)(y_1(ct) + y_1(-ct) + cy_0'(ct) - cy_0'(-ct)) \\ x_1 := z \ , \ x_2 := \dot{z} \quad (23)$$

to write down (22) in the vector matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -(T/mc) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \\ - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{m}(Q(x_1) + f(t)) \quad (24)$$

This system belongs to the more general class described by

$$\dot{x} = Ax - b\phi(c^*x) + g(t) \quad (25)$$

where x, b, c are n-vectors, A is a $n \times n$ matrix and $g : \mathbb{R} \mapsto \mathbb{R}^n$ is a n-valued vector function; the asterisk denotes transposition of vectors or matrices. For system (25) we can apply the following result [17]

Theorem 1: Consider the system (25) under the following assumptions

i) A is a Hurwitz matrix i.e. all its eigenvalues have negative real parts;

ii) the nonlinear function $\phi : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz i.e. satisfies

$$0 \leq \frac{\phi(\sigma_1) - \phi(\sigma_2)}{\sigma_1 - \sigma_2} \leq L \ , \ \forall \sigma_1, \sigma_2 \in \mathbb{R} \quad (26)$$

iii) $|f(t)| \leq M$ i.e. it is globally bounded (on $\mathbb{R}$).

If the following frequency domain inequality holds

$$\frac{1}{L} + \Re e \ H(\imath\omega) > 0 \ , \ \forall \omega \geq 0 \quad (27)$$

where $H(s) = c^*(sI - A)^{-1}b$ is the transfer function of the linear part of (25), then system (25) has a unique bounded on $\mathbb{R}$ solution which is T-periodic if $f(t)$ is such and almost periodic if $f(t)$ is such. Moreover this solution is exponentially stable.

The theorem gives a good example of synchronization with an external signal since a solution of the same kind as the external signal exists and all other solutions approach it exponentially for $t \to \infty$. Practically speaking this would be the only observable and measurable solution as stated in the *Stability postulate*.

We shall apply now Theorem 1 to (24). Observe first that the corresponding matrix A has a zero eigenvalue i.e. is in a critical case. Fortunately this difficulty can be overcome. Observe that Theorem 1 holds not just for a single given nonlinear function but for any continuous nonlinear function satisfying (26) that is for an entire class of functions. Assume that $Q(\sigma)$ is subject to a modified Lipschitz condition

$$0 < \mu \leq \frac{Q(\sigma_1) - Q(\sigma_2)}{\sigma_1 - \sigma_2} \leq L \ , \ \forall \sigma_1, \sigma_2 \in \mathbb{R} \quad (28)$$

If we introduce a new nonlinear function $\psi(\sigma) := Q(\sigma) - \mu\sigma$ then (24) becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(\mu/m) & -(T/mc) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \\ - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{m}(\psi(x_1) + f(t)) \quad (29)$$

corresponding to

$$m\ddot{z} + (T/c)\dot{z} + \mu z + \psi(z) = f(t) \quad (30)$$

It is not difficult to find here

$$H(s) = \frac{1}{ms^2 + (T/c)s + \mu} \ ;$$

$$\Re e \ H(\imath\omega) = \frac{\mu - m\omega^2}{(\mu - m\omega^2)^2 + (T/c)^2\omega^2}$$

We need

$$\frac{1}{L - \mu} + \frac{\mu - m\omega^2}{(\mu - m\omega^2)^2 + (T/c)^2\omega^2} > 0 \ ; \ \forall \omega \geq 0$$

and an elementary manipulation will give

$$L < \mu + \frac{T}{c}\left(\frac{T}{mc} + 2\sqrt{\frac{\mu}{m}}\right) \quad (31)$$

We end this section by the following remark: the character of the steady state oscillatory solution is given by the initial displacement of the string. If it is periodic then the steady state is periodic. The exponential stability of this steady state is ensured by the occurring damping and also by (28) - a strong sector condition for the nonlinear function. Observe also that (15) and (20) show that periodicity (almost periodicity) extends to the waves $v_r(x,t)$, $w_r(x,t)$, $v_l(x,t)$, $w_l(x,t)$ but they are not stable in the sense of Liapunov since the string equation is undamped. And finally: the synchronization through wave propagation may be viewed as a single-dimension version of Huygens pendula synchronization (known also as the Lord Kelvin problem - see [18].

IV. The case of the two oscillators on the string

We shall refer here to the basic reference [8] while our motivation for the model is different and has been presented in the previous sections. In this case there are two lumped parameter, nonlinear undamped oscillators "hanging" on an infinite string, at $x=-L/2$ and $x=L/2$ respectively. The equations are as follows

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2}-c^2\frac{\partial^2 y}{\partial x^2}=0\ ,\ c^2=T/\rho\ ;\ t>0\ ,\ -\infty<x<\infty\\
&m_1\ddot{z}_1+V_1(z_1)=T\left(\frac{\partial y_c}{\partial x}(-L/2,t)-\frac{\partial y_l}{\partial x}(-L/2,t)\right)\\
&m_2\ddot{z}_2+V_2(z_2)=T\left(\frac{\partial y_r}{\partial x}(L/2,t)-\frac{\partial y_c}{\partial x}(L/2,t)\right)\\
&y_l(-L/2,t)=y_c(-L/2,t)=z_1(t)\ ;\\
&y_c(L/2,t)=y_r(L/2,t)=z_2(t)\\
&z_i(0)=z_0^i\ ,\ \dot{z}_i(0)=z_1^i\ ,\ i=1,2\ ;\ y(x,0)=y_0(x)\ ,\\
&\frac{\partial y}{\partial t}(x,0)=y_1(x)\ ,\ -\infty<x<\infty
\end{aligned} \tag{32}$$

with the notations being as in the previous section where a single oscillator was concerned.

A. Starting from (32) we can consider three initial boundary value problems in the LHS $\{(x,t)|x<-L/2,t>0\}$ and RHS $\{(x,t)|x>L/2,t>0\}$ upper quadrants and in the central $\{(x,t)|-L/2<x<L/2,t>0\}$ upper central semi-strip of the plane $\mathbb{R}\times\mathbb{R}$.

We proceed as previously by defining $v(x,t)$, $w(x,t)$ as in (5) and the system in the Friedrichs form as in (6); this system is diagonalized to obtain (7) with the initial conditions (8).

B. Consider first the boundary value problem in the RHS upper quadrant which is translated by $L/2$ to the right in comparison to what we had in the previous section. Proceeding as there we obtain the representation formulae

$$\begin{aligned}
&u_r^+(x,t)=\begin{cases}
y_1(x-ct)-cy_0'(x-ct)\ ;\\
0<t<(x-L/2)/c\ ,\ x>L/2\\
2\dot{z}_2(t-(x-L/2)/c)-y_1(ct-x+L)-\\
-cy_0'(ct-x+L)\ ;t>(x-L/2)/c\ ,\\
x>L/2
\end{cases}\\
&u_r^-(x,t)=y_1(x+ct)+cy_0'(x+ct)
\end{aligned} \tag{33}$$

Using these formulae we obtain those for $v_r(x,t)$ and $w_r(x,t)$

$$\begin{aligned}
&v_r(x,t)=\begin{cases}
\frac{1}{2}[y_1(x+ct)+cy_0'(x+ct)+\\
+y_1(x-ct)+cy_0'(x-ct)]\ ;\ x-ct>L/2\ ,\\
x>L/2\\
\frac{1}{2}[y_1(x+ct)+cy_0'(x+ct)+\\
+2\dot{z}_2(t-(x-L/2)/c)-y_1(ct-x+L)-\\
-cy_0'(ct-x+L)]\ ;\\
x-ct<L/2\ ,\ x>L/2
\end{cases}\\
&w_r(x,t)=\begin{cases}
\frac{1}{2c}[y_1(x+ct)+cy_0'(x+ct)-\\
-y_1(x-ct)+cy_0'(x-ct)]\ ;\ x-ct>L/2\ ,\\
x>L/2\\
\frac{1}{2c}[y_1(x+ct)+cy_0'(x+ct)-\\
-2\dot{z}_2(t-(x-L/2)/c)+y_1(ct-x+L)+\\
+cy_0'(ct-x)]\ ;x-ct<L/2\ ,\ x>L/2
\end{cases}
\end{aligned} \tag{34}$$

In the same way, based on the approach of the previous section we obtain the representation formulae for the LHS upper quadrant

$$\begin{aligned}
&u_l^-(x,t)=\begin{cases}
y_1(x+ct)+cy_0'(x+ct)\ ;\\
x+ct<-L/2\ ,\ x<-L/2\\
2\dot{z}_1(t+(x+L/2)/c)-y_1(-L-ct-x)+\\
+cy_0'(-L-ct-x)\ ;\\
x+ct>-L/2\ ,\ x<-L/2
\end{cases}\\
&u_l^+(x,t)=y_1(x-ct)-cy_0'(x-ct)
\end{aligned} \tag{35}$$

and

$$w_l(x,t) = \begin{cases} \frac{1}{2}[y_1(x+ct) + cy_0'(x+ct) + \\ +y_1(x-ct) - cy_0'(x-ct)] \; ; \\ x+ct < -L/2 \; , \; x < -L/2 \\ \frac{1}{2}[y_1(x-ct) - cy_0'(x-ct) + \\ +2\dot{z}_1(t+(L/2+x)/c) - \\ -y_1(-ct-x-L) + cy_0'(-ct-x-L)] \; ; \\ x+ct > -L/2 \; , \; x < -L/2 \end{cases}$$

$$w_l(x,t) = \begin{cases} \frac{1}{2c}[y_1(x+ct) + cy_0'(x+ct) - \\ -y_1(x-ct) + cy_0'(x-ct)] \; ; \\ x+ct < -L/2 \; , \; x < -L/2 \\ \frac{1}{2c}[-y_1(-L-x-ct) + cy_0'(-L-x-ct) + \\ +2\dot{z}_1(t+(x+L/2)/c) - y_1(x-ct) + \\ +cy_0'(x-ct)] \; ; \\ x+ct > -L/2 \; , \; x < -L/2 \end{cases} \tag{36}$$

C. We are now in position to consider the boundary value problem in the central upper strip $\{(x,t)| - L/2 < x < L/2, t > 0\}$ defined by

$$\begin{aligned} & \frac{\partial u_c^+}{\partial t} + c\frac{\partial u_c^+}{\partial x} = 0 \; , \; \frac{\partial u_c^-}{\partial t} - c\frac{\partial u_c^-}{\partial x} = 0 \; ; \\ & -L/2 < x < L/2 \; , \; t > 0 \\ & u_c^-(-L/2,t) + u_c^+(-L/2,t) = 2\dot{z}_1(t) \; ; \\ & u_c^-(L/2,t) + u_c^+(L/2,t) = 2\dot{z}_2(t) \; , \; t > 0 \\ & u_c^+(x,0) = y_1(x) - cy_0'(x) \; , \; u_c^-(x,0) = y_1(x) + cy_0'(x) \; ; \\ & -L/2 < x < L/2 \end{aligned} \tag{37}$$

In this case the approach of the Appendix may be applied *ad litteram*. Consider some point (x,t) in the upper strip and the two characteristic lines that cross it

$$t^{\pm}(\zeta; x,t) = t \pm (\zeta - x)/c \tag{38}$$

We integrate the progressive wave from x to $L/2$ along the increasing characteristic line to obtain

$$u_c^+(x,t) = u_c^+(L/2, t + (L/2-x)/c) \tag{39}$$

and the regressive wave from $-L/2$ to x along the decreasing characteristic line to obtain

$$u_c^-(x,t) = u_c^-(-L/2, t + (L/2+x)/c) \tag{40}$$

Extend now the increasing characteristic "to the left". If this extension can be done up to $-L/2$ without leaving the strip, then

$$u_c^+(-L/2,t) = u_c^+(L/2, t+L/c) \; ; \; t > 0 \tag{41}$$

and, for the regressive wave, if the extension "to the right" is possible:

$$u_c^-(L/2,t) = u_c^-(-L/2, t+L/c) \; ; \; t > 0 \tag{42}$$

Denoting now

$$\begin{aligned} & \xi_c^+(t) := u_c^+(L/2,c) \; \Rightarrow \; u_c^+(-L/2,t) = \xi_c^+(t+L/c) \\ & \xi_c^-(t) := u_c^-(-L/2,c) \; \Rightarrow \; u_c^-(L/2,t) = \xi_c^-(t+L/c) \end{aligned} \tag{43}$$

and substituting in the boundary conditions, the following continuous time difference system is obtained

$$\begin{aligned} & \xi_c^+(t+L/c) + \xi_c^-(t) = 2\dot{z}_1(t) \\ & \xi_c^-(t+L/c) + \xi_c^+(t) = 2\dot{z}_2(t) \end{aligned} \tag{44}$$

Two remarks are useful: first, the continuous time difference equations are functional equations with deviated argument of neutral type - see e.g. the Appendix. Second, the solution of (44) can be constructed by steps provided the forcing terms $\dot{z}_i(t)$ are known and some initial conditions to start the recurrence are given. The forcing terms will result from the coupling to the ordinary differential equations of the local oscillators while the initial conditions can be obtained by integrating along those characteristics that cannot be extended to the left or to the right without leaving the strip. If the increasing characteristic can be extended up to $\zeta = x - ct$ only, then

$$\begin{aligned} & u_c^+(x-ct,0) = u_c^+(L/2, t+(L/2-x)/c) = \\ & \quad = \xi_c^+(t+(L/2-x)/c) \end{aligned} \tag{45}$$

provided $-L/2 < x - ct < L/2$. It is easily seen that if $-L/2 < x - ct < L/2$ then $0 < t + (L/2-x)/c < L/c$. Similarly

$$u_c^-(x+ct,0) = \xi_c^-(t+(L/2+x)/c) \tag{46}$$

provided $-L/2 < x + ct < L/2$. The initial conditions for (44) are thus established.

In order to couple (44) to the local oscillators we compute

$$\begin{aligned} w_c(-L/2,t) &= \frac{1}{2c}[u_c^-(-L/2,t) - u_c^+(-L/2,t)] = \\ &= \frac{1}{2c}[\xi_c^-(t) - \xi_c^+(t+L/c)] \end{aligned}$$

and make use of $w_l(-L/2,t)$ obtained from (36). It follows that

$$\begin{aligned} & m_1\ddot{z}_1 + (2T/c)\dot{z}_1 + V_1(z_1) - (T/c)\xi_c^-(t) = \\ & = (T/c)[y_1(-L/2-ct) - cy_0'(-L/2-ct)] \end{aligned} \tag{47}$$

and, similarly

$$\begin{aligned} & m_2\ddot{z}_2 + (2T/c)\dot{z}_2 + V_2(z_2) - (T/c)\xi_c^+(t) = \\ & = (T/c)[y_1(L/2+ct) + cy_0'(L/2+ct)] \end{aligned} \tag{48}$$

Making the following notations

$$\eta_c^{\pm}(t) := \xi_c^{\pm}(t + L/c) \ ;$$

$$y_1(L/2 + ct) + cy_0'(L/2 + ct) := f^+(t) \ ,$$

$$y_1(-L/2 - ct) - cy_0'(-L/2 - ct) := f^-(t)$$

we obtain the system of coupled delay differential and difference equations

$$\begin{aligned} & m_1\ddot{z}_1 + (2T/c)\dot{z}_1 + V_1(z_1) - (T/c)\eta_c^-(t - L/c) = \\ & \quad = (T/c)f^-(t) \\ & m_2\ddot{z}_2 + (2T/c)\dot{z}_2 + V_2(z_2) - (T/c)\eta_c^+(t - L/c) = \\ & \quad = (T/c)f^+(t) \\ & \eta_c^+(t) = -\eta_c^-(t - L/c) + 2\dot{z}_1(t) \\ & \eta_c^-(t) = -\eta_c^+(t - L/c) + 2\dot{z}_2(t) \end{aligned} \tag{49}$$

V. About the system of differential and difference equations

A. It can be easily seen that system (49) is of the vector matrix form

$$\begin{aligned} \dot{x} &= A_0x + A_1y(t - \tau) - \sum_1^p b_i\phi_i(c_i^*x) + f_1(t) \\ y(t) &= A_2x(t) + A_3y(t - \tau) + f_2(t) \end{aligned} \tag{50}$$

which appears in [19], [20]. Its solution can be constructed by steps on intervals $(k\tau, (k+1)\tau)$ and even if $f_2(t)$ is continuous, $y(t)$ will result only piecewise continuous, with jumps at $k\tau$, $k = 0, 1, \ldots$. It follows at once from (50) the recurrence for the jumps (discontinuities)

$$y(k\tau+0) - y(k\tau-0) = A_3(y((k-1)\tau+0) - y((k-1)\tau-0)) \tag{51}$$

If we denote by $y^o(\theta)$, $-\tau \le \theta \le 0$, the initial condition for $y(t)$, then we obtain from the second equation of (50) the following initial discontinuity at $t = 0$ which generates propagation of the discontinuities

$$y(0+) - y(0-) = A_2x_0 + A_3y^o(-\tau) + f_2(0) - y^o(0) \neq 0 \tag{52}$$

This discontinuity propagation is one of the arguments for the neutral type of system (50).

If we turn to the representation formulae (39) and (40) and make use of the notations (43) then we deduce

$$\begin{aligned} u_c^+(x,t) &= \xi_c^+(t + (L/2 - x)/c) \ , \\ u_c^-(x,t) &= \xi_c^-(t + (L/2 + x)/c) \end{aligned} \tag{53}$$

For a direct reference to the solution of (49) it can be written that

$$\begin{aligned} u_c^+(x,t) &= \eta_c^+(t - (L/2 + x)/c) \ , \\ u_c^-(x,t) &= \eta_c^-(t - (L/2 - x)/c) \end{aligned} \tag{54}$$

hence $u_c^{\pm}(x,t)$ as classical solutions will result with discontinuities on some characteristics; this is a standard property for hyperbolic PDE.

B. One might ask whether system (50) hence system (49) can have periodic or almost periodic solutions whenever the forcing terms are such. With respect to this there exists the result of [21]. In order to state this result, denote

$$\begin{aligned} H(s) &= \begin{pmatrix} sI - A_0 & -A_1\mathrm{e}^{-s\tau} \\ -A_2 & I - A_3\mathrm{e}^{-s\tau} \end{pmatrix} \ ; \\ T(s) &= \begin{pmatrix} C^* & 0 \end{pmatrix} H^{-1}(s) \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \end{aligned} \tag{55}$$

where B_1 is the matrix having b_i as columns and C^* - the one having c_i^* as rows. If x is a n-vector and y - a m-vector, the dimensions of the matrices and of the vectors are accordingly. The result, which is basic for our problem, reads as follows

Theorem 2: Consider system (50) under the following assumptions

- 1^o The eigenvalues of A_3 are inside the unit disk and $\det\ H(s) \neq 0$ for all $s \in \mathbb{C}$ such that $\Re e(s) \le -\alpha < 0$.
- 2^o The nonlinear functions $\phi_i : \mathbb{R} \mapsto \mathbb{R}$ are globally Lipschitz i.e. subject to

$$\begin{aligned} & 0 \le \frac{\phi_i(\sigma_1) - \phi_i(\sigma_2)}{\sigma_1 - \sigma_2} \le L_i \ , \ \forall \sigma_1, \sigma_2 \in \mathbb{R} \ , \\ & \phi_i(0) = 0 \ ; \ i = \overline{1,m} \end{aligned} \tag{56}$$

- 3^o There exist $\tau_i > 0$ and some $\delta > 0$ such that

$$\tau_d L_d^{-1} + \Re e\ \tau_d T(\imath\omega) \ge \delta I \ , \ \forall \omega > 0 \tag{57}$$

  where τ_d and L_d are diagonal matrices having on the main diagonal the positive real numbers $\tau_i > 0$ and $L_i > 0$ respectively.
- 4^o $|f_1(t)| + |f_2(t)| \le M$ for all $t \in \mathbb{R}$

Then there exists a bounded on $\mathbb{R}$ solution of (50) which is periodic or almost periodic respectively if f_1 and f_2 are periodic or almost periodic respectively. Moreover *this global solution* is exponentially stable.

Here $\Re e\ G(\imath\omega)$ means $(1/2)[G(\imath\omega) + G^*(\imath\omega))$ where the star denotes transpose and complex conjugate; for two symmetric matrices G_1, G_2 the inequality $G_1 \ge G_2$ means that $G_1 - G_2 \ge 0$ is a positive definite matrix.

B. We shall apply Theorem 2 to system (4.14). This means checking fulfilment of Theorem's assumptions by the coefficients of system (49). The easiest to check is $1^o(a)$ which signifies strong stability of the difference operator in the system of neutral functional differential equations [20]. But it is obvious from (49) that the eigenvalues of the corresponding matrix A_3 are ± 1 hence the difference operator is only critically stable. This is not surprising, according to our experience e.g. [15] - all systems with propagation from Mechanics are such. Unfortunately this un-fulfilment is far going:strong stability of the difference operator is usually a necessary condition for the exponential stability of the linear part of (49) i.e. of the neutral system

$$\begin{aligned} &m_1\ddot{z}_1 + (2T/c)\dot{z}_1 - (T/c)\eta_c^-(t - L/c) = 0 \\ &m_2\ddot{z}_2 + (2T/c)\dot{z}_2 - (T/c)\eta_c^+(t - L/c) = 0 \\ &\eta_c^+(t) + \eta_c^-(t - L/c) - 2\dot{z}_1(t) = 0 \\ &\eta_c^-(t) + \eta_c^+(t - L/c) - 2\dot{z}_2(t) = 0 \end{aligned} \tag{58}$$

Its characteristic equation has the form (obtained after some elementary manipulation

$$\begin{aligned} H(s) = s^2 \left[(m_1 s + 2T/c)(m_2 s + 2T/c) - \right. \\ \left. - m_1 m_2 s^2 \mathrm{e}^{-2sL/c}\right] = 0 \end{aligned} \tag{59}$$

Obviously this equation has a double zero root. Since system (49) has two nonlinear functions, this corresponds to the first critical case in the theory of the absolute stability. For the stability of the zero solution this might not be a problem but, according to our knowledge, there are no existence results for nonlinear forced oscillations to hold in critical cases.

As in the case of a single oscillator (previous section), it is helpful to make assumptions of the form (28) for $V_i(z_i)$. Under these circumstances system (58) will take the form

$$\begin{aligned} &m_1\ddot{z}_1 + (2T/c)\dot{z}_1 + \mu z_1 - (T/c)\eta_c^-(t - L/c) = 0 \\ &m_2\ddot{z}_2 + (2T/c)\dot{z}_2 + \mu z_2 - (T/c)\eta_c^+(t - L/c) = 0 \\ &\eta_c^+(t) + \eta_c^-(t - L/c) - 2\dot{z}_1(t) = 0 \\ &\eta_c^-(t) + \eta_c^+(t - L/c) - 2\dot{z}_2(t) = 0 \end{aligned} \tag{60}$$

what will lead to the following characteristic equation

$$\begin{aligned} &H(s) = (m_1 s^2 + 2(T/c)s + \mu)(m_2 s^2 + 2(T/c)s + \mu) - \\ &-(m_1 s^2 + \mu)(m_2 s^2 + \mu)\mathrm{e}^{-2sL/c} = 0 \end{aligned} \tag{61}$$

As shows our previous experience, it is possible that H(s) defined by (61) has its roots in some left half-plane $\Re e(s) \leq -\alpha < 0$ in spite of the fact that the eigenvalues of A_3, namely $\lambda = \pm 1$ are on the unit circle and not inside the unit disk. It is not impossible to have the frequency domain inequality (57) also fulfilled for system (49). The proof of Theorem 2 is nevertheless dependent, in some of its points, of the location of the eigenvalues of A_3 inside the unit disk and *not* on its boundary - the unit circle.

We thus consider this application as a challenge in order to extend the theory of the neutral functional differential equations by relaxing some of its basic assumptions.

VI. Conclusions and open problems

A. We have discussed in this paper some qualitative problems connected to the oscillatory behavior of local oscillators mutually coupled through a vibrating string. Instead of considering e.g. "incident waves coming from $-\infty$" [7] we embedded the oscillatory excitation in the initial conditions of the vibrating string. In our opinion this is a better model validation since the phenomenon is completely described by the boundary value problem and its initial conditions. Within this framework the local oscillators appear as externally forced by the initial conditions of the vibrating string and if these initial conditions (that generate waves) are periodic or almost periodic, the steady state solutions of the local oscillators may be such, provided some existence theorems from Oscillation Theory are valid. In the case of a single oscillator, this theorem, due to Yakubovich [17], holds if the nonlinear function is subject to a certain inequality given by (28). The fact that under these assumptions the local oscillator oscillates periodically if the forcing waves generated by the string initial displacements are periodic or almost periodically if the forcing waves are such, is obviously some form of synchronization. Worth mentioning that in the periodic case the period is the same for both forcing and forced oscillations even if the presence of the nonlinearity induces higher harmonics.

B. The case of the two local oscillators is more interesting since it generates functional differential equations of neutral type with all problems associated to such mathematical objects. Leaving aside the discontinuities of the solutions, the main difficulty is here the strong stability of the difference operator. As already mentioned, the equations arising from Mechanics do not generate, generally speaking, strongly stable but only critically stable difference operators. This aspect requires a basic reformulation of the existence theorems and new mathematical proofs. Obtaining existence and stability of forced oscillations in critical cases would be an important step forward in this field.

On the other hand, synchronization of two local oscillators connected by a string may be viewed as a simplified model for the basic problem of Huygens, where two pendula got synchronized through a medium with distributed parameters. According to R. W. Brockett [18] synchronization may be achieved, in the classical case of Huygens also, by introducing a nonlinear feedback integral control. This approach - the control theoretic approach - represents another direction in synchronization, which is viewed as a servomechanism (tracking) problem in some structure of oscillators.

C. We cannot end this paper without pointing out another direction of development which is important in applications. The so called *consensus over networks of agents* is modeled by independent, even identical dynamical systems (called by Lagrange - oscillators) which are connected in some graph structure. The links may be quite complex - we mentioned some aspects in the Introduction - but here we just recall that one can also discover synchronization and systems with complex links in this last case also. And last but not least - mechanical models may be replaced by electrical ones or even other - suggested by applications in Physics, Biology, Engineering.

D. The present paper represent the initial - extended - version of a paper presented in an invited session at the International Conference on Numerical Analysis Analysis 2015 taking place in Rhodes in September 2015. It was published under a rather shorten version in [22]. It is felt this basic form could be stimulating for further development, especially if recent results were to be taken into account. Citing all connected references would sensibly increase their list. As always the hopes go towards the interested readers.

VII. Appendix

We reproduce here, for the sake of completeness, the basic theorem of Cooke [14] as it is stated and completely proved in [15]. This theorem is the methodological basis for all development of this paper. The first mathematical object to be considered is the following boundary value problem with initial and derivative boundary conditions

$$\begin{aligned} &\frac{\partial u^+}{\partial t} + \tau^+(\lambda,t)\frac{\partial u^+}{\partial \lambda} = \Phi^+(\lambda,t) \\ &\frac{\partial u^-}{\partial t} + \tau^-(\lambda,t)\frac{\partial u^-}{\partial \lambda} = \Phi^-(\lambda,t),\ 0\le\lambda\le 1,\ t\ge t_0, \\ &\sum_{k=0}^{m}\left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(0,t) + a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(0,t)\right] = f_0(t) \\ &\sum_{k=0}^{m}\left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^+(1,t) + b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}u^-(1,t)\right] = f_1(t) \\ &u^\pm(\lambda,t_0) = \omega^\pm(\lambda)\ ,\ 0\le\lambda\le 1 \end{aligned} \tag{62}$$

with $\tau^+(\lambda,t)>0$, $\tau^-(\lambda,t)<0$. Consider the two families of characteristics

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{\tau^\pm(\lambda,t)},\ \tau^+(\lambda,t)>0\ ,\ \tau^-(\lambda,t)<0 \tag{63}$$

and let $t^\pm(\sigma;\lambda,t)$ the two characteristic curves crossing some point (λ,t) of the strip $[0,1]\times[t_0,t_1)$. Define

$$T^+(t) := t^+(1;0,t) - t\ ,\ T^-(t) := t^-(0;1,t) - t \tag{64}$$

called *propagation times along the characteristics* or *forward and backward* propagation time respectively.

The other mathematical object is the following system of differential equations with deviated arguments

$$\begin{aligned} &\sum_{k=0}^{m}\left[a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^+(t+T^+(t)) + a_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^-(t)\right] = \\ &= f_0(t) + \sum_{k=0}^{m}a_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Psi^+(t) \\ &\sum_{k=0}^{m}\left[b_k^+(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^+(t) + b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}y^-(t+T^-(t))\right] = \\ &= f_1(t) - \sum_{k=0}^{m}b_k^-(t)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Psi^-(t) \end{aligned} \tag{65}$$

whose solutions can be constructed by steps for $t > t_0 + \max\{T^-(t_0),T^+(t_0)\}$ provided some initial conditions are given on $[t_0, t_0+\max\{T^-(t_0),T^+(t_0)\}]$. We denoted

$$\begin{aligned} \Psi^+(t) &:= \int_0^1 \frac{\Phi^+(\sigma,t^+(\sigma;0,t))}{\tau^+(\sigma,t^+(\sigma;0,t))}\mathrm{d}\sigma\ ; \\ \Psi^-(t) &:= \int_0^1 \frac{\Phi^-(\sigma,t^-(\sigma;1,t))}{\tau^-(\sigma,t^-(\sigma;1,t))}\mathrm{d}\sigma \end{aligned} \tag{66}$$

We may now state

Theorem 3: Consider the boundary value problem (62) and let $u^\pm(\lambda,t)$ be a (possibly discontinuous) classical solution of it. Let $y^+(t) := u^+(1,t)$, $y^-(t) := u^-(0,t)$. These functions define a solution of (65) with the initial conditions

$$y_0^+(t^+(1;\lambda,t_0)) = \omega^+(\lambda) + \int_\lambda^1 \frac{\Phi^+(\sigma,t^+(\sigma;\lambda,t_0))}{\tau^+(\sigma,t^+(\sigma;\lambda,t_0))}\mathrm{d}\sigma \tag{67}$$

where $0\le\lambda\le 1\ \Leftrightarrow\ t_0 \le t^+(1;\lambda,t_0) \le t_0 + T^+(t_0)$ and

$$y_0^-(t^-(0;\lambda,t_0)) = \omega^-(\lambda) - \int_0^\lambda \frac{\Phi^-(\sigma,t^-(\sigma;\lambda,t_0))}{\tau^-(\sigma,t^-(\sigma;\lambda,t_0))}\mathrm{d}\sigma \tag{68}$$

where $0\le\lambda\le 1\ \Leftrightarrow\ t_0 \le t^-(0;\lambda,t_0) \le t_0 + T^-(t_0)$.

Conversely, let $y^\pm(t)$ be a sufficiently smooth solution of (65) with some initial conditions $y_0^\pm(t)$ defined on $[t_0, t_0+\max\{T^-(t_0),T^+(t_0)\}]$. Then $u^\pm(\lambda,t)$ defined by

$$\begin{aligned} u^+(\lambda,t) &= y^+(t^+(1;\lambda,t)) - \int_\lambda^1 \frac{\Phi^+(\sigma,t^+(\sigma;\lambda,t))}{\tau^+(\sigma,t^+(\sigma;\lambda,t))}\mathrm{d}\sigma \\ u^-(\lambda,t) &= y^-(t^-(0;\lambda,t)) + \int_0^\lambda \frac{\Phi^-(\sigma,t^-(\sigma;\lambda,t))}{\tau^-(\sigma,t^-(\sigma;\lambda,t))}d\sigma \end{aligned} \tag{69}$$

is a solution of (62) with the initial condition $\omega^\pm(\lambda)$ computed also from (69) at $t=t_0$

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