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Around the dynamics and control of a model arising from vibroimpact mechanics

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***Abstract*—It is considered a simple model of contact mechanics arising from vibroimpact machinery modeling. This model is deduced using the variational principle of Hamilton applied to a structure with distributed parameters. Its novelty includes introduction of the elastic strain induced by the external force at the boundary. The non-penetrating contact is modeled by the Hertz-Signorini-Moreau complementarity condition - again in the simplest case. Next, there is studied asymptotic stability of the autonomous system, based on the energy Lyapunov functional and the Barbashin- Krasovskii-LaSalle invariance principle.**

***Index Terms*—vibroimpact mechanics,complementarity condition,stability,energy Lyapunov functional, Barbashin-Krasovskii-LaSalle invariance principle.**

I. Introduction and problem statement

Vibroimpact mechanical machinery is of utmost importance e.g. in civil and mining engineering [1]. The reader is sent to the aforementioned classical reference on its theory, design and various applications. It has to be mentioned also that vibroimpact machinery theory and applications turned to be an interesting motivation for the development of contact and non-smooth Mechanics [2]–[4]. Such systems clearly generate complex oscillatory behavior as a consequence of the complexity of the models.

On the other hand it is an elementary fact that modeling of the physical systems starts with some kind of decomposition in simpler subsystems which are modeled separately and interconnected afterwards. Even analysis and modeling of such simpler subsystems can turn into a non-easy job.

In this paper we consider the model of a beam with distributed parameters in linear motion under a driving force at one boundary. At the other boundary a non-penetrating contact is present, introducing reverse motion and other phenomena modeled by non-smooth and/or complementarity systems. We are thus led to differential equations with discontinuous R(ight) H(and) S(ide) with their various definitions of the solutions.

The novelty element in this paper is given by the integration of the elastic strain in the mechanical model: the driving force moves the mobile part of the system but also produces elastic strain. Such "strain losses" will turn to have a stabilizing effect. From the theoretical point of view this model completion will allow the use of the "weak" energy Lyapunov functional combined with the application of the Barbashin-Krasovskii-LaSalle invariance principle to obtain asymptotic stability.

Summarizing, what is left of the paper is structured as follows. The model is obtained starting from the Hamilton variational principle [5]–[7] for systems with distributed parameters (in this case - beams in linear motion). The key elements of the approach are given by the choice of the forces acting on the system together with their expression on the generalized coordinates. To the model - an I(nitial) B(oundary) V(alue) P(roblem) with derivative boundary conditions - it is associated the energy identity. A simple manipulation of the energy identity and of the boundary conditions will lead to the energy Lyapunov functional giving the Lyapunov stability in the metrics induced by the Lyapunov functional itself.

For the asymptotic stability we apply the methodology from our survey [8]: we associate to the IBVP a system of F(unctional) D(ifferential) E(quation) with deviated argument whose solutions are in one-to-one correspondence with the solution of the IBVP. Applying the Barbashin-Krasovskii-LaSalle invariance principle to the system of FDE, its asymptotic stability is obtained. Due to the one-to-one correspondence of the solutions of the two mathematical objects, asymptotic stability is projected back on the solutions of the IBVP.

In the final - Conclusions section - there is discussed the role of the model for the impact force as well as of the elastic strain in establishing the stability for system's equilibrium.

II. The variational modeling

Consider the structure of figure 1

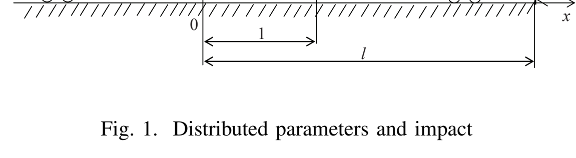

Fig. 1. Distributed parameters and impact

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2.1 We shall apply the variational principle of Hamilton in order to obtain the model. The Hamilton functional reads

$$I(t_1,t_2)=\int_{t_1}^{t_2}(E_k(t)-E_p(t)+W_m(t))\mathrm{d}t \tag{1}$$

In (1) we have to express the kinetic energy $E_k(t)$, the potential energy $E_p(t)$ and the mechanical work $W_m(t)$ associated to various forces. The kinetic energy $E_k(t)$ is given by

$$\begin{aligned}E_k(t)=\frac{1}{2}\Big\{&m_0\dot{z}_1^2(t)+m_1u_t^2(l,t)+\\&+\int_0^l\rho(x)\Gamma(x)u_t^2(x,t)\mathrm{d}x\Big\}\end{aligned} \tag{2}$$

Here $m_i, i=0,1$ are lumped masses at system's boundaries, $\rho(x)$ is beam's mass density per beam's length and $\Gamma(x)$ - the cross-section area of the beam at the current coordinate $x\in(0,l)$. It has to be mentioned that $u(x,t)$ - the linear displacement of the beam - incorporates the elastic strain $\zeta(x,t)$ also i.e. $u(x,t)=z_1(t)+\zeta(x,t)$.

The potential energy which is stored in the elastic strain is given by

$$\begin{aligned}E_p(t)&=\frac{1}{2}\int_0^l\Gamma(x)E(x)\zeta_x^2(x,t)\mathrm{d}x=\\&=\frac{1}{2}\int_0^l\Gamma(x)E(x)u_x^2(x,t)\mathrm{d}x\end{aligned} \tag{3}$$

Here $E(x)$ is the Young elasticity modulus of the beam.

To express the work $W_m(t)$ we need a "list" of the various forces acting on the system as follows

- the local restoring force of the driving mechanism

$$f_r(t)=-k_0z_1(t)-c_0\dot{z}_1(t) \tag{4}$$

  composed of an elastic and a viscous damping components, $z_1(t)$ being the displacement imposed by the external active "pushing" force $f_0(t)$ - see figure 1;
- the active force $f_0(t)$ and the distributed external active force $f_a(x,t)$;
- the active force $f_0'(t)=c_0'\dot{z}_1(t)$ effectively transmitted to the load;
- the load (from the "point of view" of the driving mechanism) at the mechanical load

$$f_0''(t)=-c_0'u_t(0,t) \tag{5}$$

  Observe that f_0' and f_0'' are virtual forces arising from the separation of the driving system from the load; they might be equal in the case of the perfect stiffness - zero elastic strain;
- the viscous distributed damping force within the beam

$$f_d(x,t)=-c(x)\Gamma(x)u_t(x,t) \tag{6}$$

  with $c(x)$ being the distributed damping coefficient;
- the "wall" reaction at the non-penetrating impact $-f_1(t)$ - see again figure 1;
- the local restoring force at $x=l$

$$f_r'(t)=-k_1u(l,t)-c_1u_t(l,t) \tag{7}$$

We are now in position to express the mechanical work of the aforementioned forces as follows

$$\begin{aligned}W_m(t)&=(f_0(t)+f_r(t)+f_0''(t))z_1(t)+\\&+(f_0'(t)+f_0''(t))u(0,t)+(f_r'(t)-f_1(t))u(l,t)+\\&+\int_0^l(f_a(x,t)+f_d(x,t))u(x,t)\mathrm{d}x\end{aligned} \tag{8}$$

2.2 Consider now the Euler Lagrange variations of the generalized coordinates with respect to a solution considered to ensure the extremum of the functional

$$u(x,t)=\bar{u}(x,t)+\varepsilon\varsigma(x,t)\ ,\ z_1(t)=\bar{z}_1(t)+\varepsilon\zeta_1(t) \tag{9}$$

The Hamiltonian functional reads

$$\begin{aligned}I^\varepsilon(t_1,t_2)&=\frac{1}{2}\int_{t_1}^{t_2}\Big[m_0(\dot{\bar{z}}_1(t)+\varepsilon\zeta_1(t))^2+\\&+m_1(\bar{u}_t(l,t)+\varepsilon\varsigma_t(l,t))^2+\\&+\int_0^l\rho(x)\Gamma(x)(\bar{u}_t(x,t)+\varepsilon\varsigma_t(x,t))^2\mathrm{d}x\Big]\mathrm{d}t-\\&-\frac{1}{2}\int_{t_1}^{t_2}\int_0^l E(x)\Gamma(x)(\bar{u}_x(x,t)+\varepsilon\varsigma_x(x,t))^2\mathrm{d}x\mathrm{d}t+\\&+\int_{t_1}^{t_2}[(f_0(t)+f_r(t)+f_0''(t))(\bar{z}_1(t)+\varepsilon\zeta_1(t))^2+\\&+(f_0'(t)+f_0''(t))(\bar{u}(0,t)+\varepsilon\varsigma(0,t))+\\&+(f_r'(t)-f_1(t))(\bar{u}(l,t)+\varepsilon\varsigma(l,t))+\\&+\int_0^l(f_a(x,t)+f_d(x,t))\bar{u}((x,t)+\varepsilon\varsigma(x,t))\mathrm{d}x\Big]\mathrm{d}t\end{aligned} \tag{10}$$

The functional I^ε is quadratic in ε hence it has a unique extremum. The term in ε^2 being positive, the extremum is a minimum. This unique minimum is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}I^\varepsilon(t_1,t_2)|_{\varepsilon=0}=0 \tag{11}$$

The condition (11) is leading to the equations of the solutions ensuring this minimum. Using the integration by parts, the Fubini theorem and the fact that the Euler-lagrange variations vanish at $t=t_1$, $t=t_2$, the following system is obtained

$$\begin{aligned}&\rho(x)\Gamma(x)u_{tt}+c(x)\Gamma(x)u_t-(E(x)\Gamma(x)u_x)_x=f_a(x,t)\\&m_0\ddot{z}_1+c_0\dot{z}_1+k_0z_1+c_0'u_t(0,t)=f_0(t)\\&E(0)\Gamma(0)u_x(0,t)+c_0'(\dot{z}_1(t)-u_t(0,t))=0\\&m_1u_{tt}(l,t)+c_1u_t(l,t)+k_1u(l,t)+\\&+E(l)\Gamma(l)u_x(l,t)=-f_1(t)\end{aligned} \tag{12}$$

Equations (12) need to be completed by the so called complementarity conditions which describe the impact with the "wall" (obstacle) at $x=l$. The usual condition is the so called Hertz Signorini Moreau contact-complementarity condition

$$0\le f_1(t)\perp(l-u(l,t))+\gamma_1u_t(l,t)\ge 0 \tag{13}$$

In the next section of the paper we shall discuss certain issues concerning (13).

III. The complementarity Hertz Signorini Moreau condition

Following [4], condition (13) can be translated as

- the standard position condition

$$f_1(t) \geq 0 \; ; \; z_2(t) - l \leq 0 \; , \; (l - z_2(t))f_1(t) = 0 \quad (14)$$

- the standard velocity condition

$$f_1(t) \geq 0 \; ; \; \dot{z}_2(t) \leq 0 \; , \; \dot{z}_2(t)f_1(t) = 0 \quad (15)$$

- the more general linear combination of the position and velocity conditions

$$\begin{aligned} & f_1(t) \geq 0 \; ; \; l - z_2(t) - \gamma_1 \dot{z}_1(t) \geq 0 \; , \\ & (l - z_2(t) - \gamma_1 \dot{z}_1(t))f_1(t) = 0 \end{aligned} \quad (16)$$

with $\gamma_1 > 0$.

In the following we shall focus on (15) and write down a possible solution for the inequalities of (15)

$$f_1 = \begin{cases} 0 & \text{if } \dot{z}_2 \geq 0 \\ \lambda \dot{z}_2 & \text{if } \dot{z}_2 \leq 0 \; , \; \lambda > 0 \end{cases} \quad (17)$$

Observe that (17) define a sector restricted function

$$0 \leq f_1(\sigma)\sigma \leq \lambda \sigma^2 \quad (18)$$

(see figure 2)

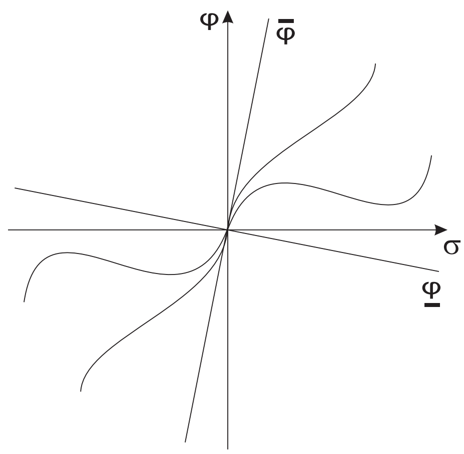

Fig. 2. Sector restricted nonlinearity.

Introduction of the sector restricted functions will allow to obtain conditions of qualitative behavior (stability, stable forced oscillations) by applying the methods of the absolute stability [9], [10]. These conditions will give results which are valid for an entire class of systems: to each nonlinearity of the class a corresponding system can be attached.

IV. Constant steady states and inherent stability

Throughout this paper we shall understand by steady states those system trajectories which *are not* defined by initial conditions but are, mathematically speaking, solutions defined on the whole real axis. We include here equilibria (which are constant steady states) but also periodic, almost periodic or even stochastic oscillations.

In order to discuss constant steady states of (12) we shall let to zero the time varying forcing terms $f_a(x,t)$ and $f_0(t)$. Since we adopted for $f_1(t)$ the form (17), we cannot consider it as forcing. Letting all time derivatives to 0 and taking $z_2(t) \equiv u(l,t)$ - as we already did in (14)-(17) - the following steady state equations are obtained from (12)

$$\begin{aligned} & (E(x)\Gamma(x)\bar{u}_x(x))_x = 0 \; ; \; k_0 \bar{z}_1 = k_0 \bar{u}(l) = 0 \\ & E(0)\Gamma(0)\bar{u}_x(0) = 0 \\ & k_1 \bar{u}(l) + E(l)\Gamma(l)\bar{u}_x(l) = 0 \end{aligned} \quad (19)$$

A straightforward manipulation will give the zero equilibrium

$$\bar{z}_1 = \bar{z}_2 = 0 \; , \; \bar{u}(x) \equiv 0 \; , \; 0 \leq x \leq l \quad (20)$$

According to the Stability Postulate of N. G. Četaev, only stable steady states are observable and measurable. Therefore it is important to check the inherent stability of the equilibrium at 0. We shall start from the energy identity: multiplying the first equation of (12) by $u_t(x,t)$ we obtain

$$\begin{aligned} & \rho(x)\Gamma(x)u_{tt}u_t + c(x)\Gamma(x)u_t^2 - (E(x)\Gamma(x)u_x)_x u_t = \\ & \qquad = f_a(x,t)u_t \end{aligned}$$

To the above inequality we add the identity

$$\begin{aligned} & (E(x)\Gamma(x)u_x)_{tx} - \frac{1}{E(x)\Gamma(x)}(E(x)\Gamma(x)u_x)\times \\ & \qquad \times(E(x)\Gamma(x)u_x)_t \equiv 0 \end{aligned}$$

thus obtaining

$$\begin{aligned} & \frac{1}{2}\Gamma(x)\frac{\mathrm{d}}{\mathrm{d}t}[\rho(x)u_t^2(x,t) + E(x)u_x^2(x,t)] - \\ & -(E(x)\Gamma(x)u_x(x,t)u_t(x,t))_x \equiv f_a(x,t)u_t(x,t) \end{aligned} \quad (21)$$

Integrating (21) with respect to x on $(0,l)$ and taking into account the boundary conditions in (12), the following identity is obtained

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{k_0 z_1(t)^2 + m_0 \dot{z}_1(t)^2 + k_1 z_2(t)^2 + m_1 \dot{z}_2(t)^2 + \\ & + \int_0^l \Gamma(x)(\rho(x)u_t^2(x,t) + E(x)u_x^2(x,t))\mathrm{d}x\Big\} + \\ & + c_0 \dot{z}_1^2(t) + c_1 \dot{z}_2^2(t) + f_1(\dot{z}_2(t))\dot{z}_2(t) + \\ & + \int_0^l c(x)\Gamma(x)u_t^2(x,t)\mathrm{d}x \equiv f_0(t)\dot{z}_1(t) + \\ & + \int_0^l f_a(x,t)u_t(x,t)\mathrm{d}x \end{aligned} \quad (22)$$

For the inherent stability of the zero equilibrium of (12) we let again the external forces go to zero in (12) and in (22) also, to observe that (22) suggests the following *energy Lyapunov functional* - written along the solutions of (12) with $z_2(t) \equiv u(l,t)$

$$\begin{aligned}
&\mathcal{V}^\star(t) \equiv \mathcal{V}(z_1(t), z_2(t), \dot z_1(t), \dot z_2(t), u_t(\cdot,t), u_x(\cdot,t)) \equiv \\
&\equiv \frac{1}{2}\{k_0 z_1^2(t) + m_0 \dot z_1^2(t) + k_1 z_2^2(t) + m_1 \dot z_2^2(t) + \\
&+ \int_0^l \Gamma(x)[\rho(x) u_t^2(x,t) + E(x) u_x^2(x,t)]\mathrm{d}x\}
\end{aligned} \tag{23}$$

Then identity (22) becomes - remind that $f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$ -

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^\star(t) + c_0 \dot z_1^2(t) + c_1 \dot z_2^2(t) + f_1(\dot z_2(t))\dot z_2(t) + \\
&\qquad + \int_0^l c(x)\Gamma(x) u_t^2(x,t)\mathrm{d}x \equiv 0
\end{aligned} \tag{24}$$

The quadratic terms in the LHS of (24) represent damping terms of the viscous friction (both lumped and distributed); due to the sector condition (18) the force f_1 also has a dissipative role.

We deduce from (24) that the Lyapunov functional (23) is non-increasing along system's solutions - in the autonomous case - hence the equilibrium is Lyapunov stable in the sense of the metrics induced by the energy Lyapunov functional itself.

As pointed out by the classics of the stability theory e.g. [11], [12], the energy Lyapunov function(al) is a weak one, in the sense that its derivative, being only non-increasing, does not meet the requirements of the Lyapunov theorem on asymptotic stability,

Therefore we have to obtain the inherent asymptotic stability by using the Barbashin-Krasovskii- LaSalle invariance principle for dynamic systems like (12). The approach of the present paper - described in the next section - relies on the Barbashin-Krasovskii-LaSalle invariance principle applied to a system of differential equations with deviated argument. This system is associated to (12) and a one-to-one correspondence is established between the solutions of the two mathematical objects. Consequently each result obtained for one mathematical object is projected back on the other.

V. The System of Equations with Deviated Argument

5.1 We start by introducing new variables in order to re-write (12) in the so called t-hyperbolic Friedrichs form [13], p. 88

$$u_t(x,t) := v(x,t)\ ,\ E(x)\Gamma(x)u_x(x,t) := w(x,t) \tag{25}$$

and also $z_2(t) := u(l,t)$. Therefore (7) become

$$\begin{aligned}
&\rho(x)\Gamma(x)v_t - w_x + c(x)\Gamma(x)v = f_a(x,t) \\
&w_t - E(x)\Gamma(x)v_x = 0 \\
&m_0\ddot z_1 + c_0\dot z_1 + k_0 z_1 + c_0' v(0,t) = f_0(t) \\
&w(0,t) + c_0'(\dot z_1(t) - v(0,t)) = 0\ ;\ \dot z_2 = v(l,t) \\
&m_1\ddot z_2 + c_1\dot z_2 + k_1 z_2 + w(l,t) = -f_2(\dot z_2)
\end{aligned} \tag{26}$$

Observe again that the new variable $z_2(t) := u(l,t)$ was already used in the conditions (14)-(16) defining the force $f_2(t)$.

We introduce next the Riemann invariants of the problem (26) as follows

$$\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{2}}\left[r^+(x,t) + \frac{1}{a(x)}r^-(x,t)\right] \\
w(x,t) &= \frac{1}{\sqrt{2}}[-a(x)r^+(x,t) + r^-(x,t)]
\end{aligned} \tag{27}$$

and the converse relations

$$\begin{aligned}
r^+(x,t) &= \frac{1}{\sqrt{2}}\left[v(x,t) - \frac{1}{a(x)}w(x,t)\right] \\
r^-(x,t) &= \frac{1}{\sqrt{2}}[a(x)v(x,t) + w(x,t)]
\end{aligned} \tag{28}$$

where we denoted $a(x) : \Gamma(x)\sqrt{\rho(x)E(x)}$. We re-write (26) in the Riemann invariants

$$\begin{aligned}
&r_t^+ + \sqrt{E(x)/\rho(x)}r_x^+ + \frac{1}{2\rho(x)}\left(\frac{a'(x)}{\Gamma(x)} + c(x)\right)r^+ + \\
&+\frac{1}{2\rho(x)a(x)}\left(-\frac{a'(x)}{\Gamma(x)} + c(x)\right)r^- = \frac{1}{\sqrt{2}}f_a(x,t) \\
&r_t^- - \sqrt{E(x)/\rho(x)}r_x^- + \frac{a(x)}{2\rho(x)}\left(\frac{a'(x)}{\Gamma(x)} + c(x)\right)r^+ + \\
&+\frac{1}{2\rho(x)}\left(\frac{a'(x)}{\Gamma(x)} + c(x)\right)r^- = \frac{1}{\sqrt{2}}a(x)f_a(x,t) \\
&m_0\ddot z_1 + c_0\dot z_1 + k_0 z_1 + \\
&\qquad +\frac{c_0'}{\sqrt{2}}\left(r^+(0,t) + \frac{1}{a(0)}r^-(0,t)\right) = f_0(t) \\
&\frac{1}{\sqrt{2}}[-a(0)r^+(0,t) + r^-(0,t)] + \\
&+c_0'\left[\dot z_1(t) - \frac{1}{\sqrt{2}}\left(r^+(0,t) + \frac{1}{a(0)}r^-(0,t)\right)\right] = 0 \\
&\dot z_2 = \frac{1}{\sqrt{2}}\left(r^+(l,t) + \frac{1}{a(l)}r^-(l,t)\right) \\
&m_1\ddot z_2 + c_1\dot z_2 + k_1 z_2 + \\
&\qquad +\frac{1}{\sqrt{2}}(-a(l)r^+(l,t) + r^-(l,t)) = -f_1(\dot z_2)
\end{aligned} \tag{29}$$

5.2 In order to simplify further development we assume homogeneous material for the beam (constant parameters) and also negligible internal damping of the beam, thus obtaining the decoupling of the Riemann invariants. Equations (29) become

$$\begin{aligned}
&r_t^+ + \sqrt{E/\rho}r_x^+ = \frac{1}{\sqrt{2}}f_a(x,t)\\
&r_t^- - \sqrt{E/\rho}r_x^- = \frac{1}{\sqrt{2}}af_a(x,t)\ (a=\Gamma\sqrt{\rho E})\\
&m_0\ddot{z}_1 + c_0\dot{z}_1 + k_0 z_1 +\\
&\quad + \frac{c_0'}{\sqrt{2}}\left(r^+(0,t) + \frac{1}{a}r^-(0,t)\right) = f_0(t)\\
&-(c_0'+a)r^+(0,t) + (1-c_0'/a)r^-(0,t) + c_0'\sqrt{2}\dot{z}_1 = 0\\
&r^-(l,t) + ar^+(l,t) - a\sqrt{2}\dot{z}_2 = 0\\
&m_1\ddot{z}_2 + c_1\dot{z}_2 + k_1 z_2 +\\
&\quad + \frac{1}{\sqrt{2}}(r^+(0,t) - ar^-(0,t)) = -f_1(\dot{z}_2)
\end{aligned} \tag{30}$$

Consider now the two families of characteristics of (30) that is

$$t^\pm(\sigma;x,t) = t \pm (\sigma - x)\sqrt{\rho/E} \tag{31}$$

and the Riemann invariants along the characteristics crossing some point $(x,t)\in(0,l)\times\mathbb{R}^+$ (fixed for a while)

$$\varphi^\pm(\sigma) := r^\pm(\sigma, t\pm(\sigma-x)\sqrt{\rho/E}) \tag{32}$$

It follows that

$$\begin{aligned}
\frac{d\varphi^+}{d\sigma} &= r_x^+(\sigma, t+(\sigma-x)\sqrt{\rho/E}) +\\
&+ \sqrt{\rho/E}r_t^+(\sigma, t+(\sigma-x)\sqrt{\rho/E}) =\\
&= \frac{1}{\sqrt{2}}\sqrt{\rho/E}f_a(\sigma, t+(\sigma-x)\sqrt{\rho/E})\\
\frac{d\varphi^-}{d\sigma} &= r_x^-(\sigma, t-(\sigma-x)\sqrt{\rho/E}) -\\
&- \sqrt{\rho/E}r_t^-(\sigma, t-(\sigma-x)\sqrt{\rho/E}) =\\
&= \frac{a}{2}\sqrt{\rho/E}f_a(\sigma, t-(\sigma-x)\sqrt{\rho/E})
\end{aligned} \tag{33}$$

We integrate (33) from l to x and from 0 to x respectively to obtain

$$\begin{aligned}
\varphi^+(x) &= r^+(x,t) = r^+(l, t+(l-x)\sqrt{\rho/E}) -\\
&- \sqrt{\frac{\rho}{2E}}\int_x^l f_a(\sigma, t+(\sigma-x)\sqrt{\rho/E})d\sigma\\
\varphi^-(x) &= r^-(x,t) = r^-(0, t+x\sqrt{\rho/E}) +\\
&+ \frac{a}{2}\sqrt{\frac{\rho}{E}}\int_0^x f_a(\sigma, t-(\sigma-x)\sqrt{\rho/E})d\sigma
\end{aligned} \tag{34}$$

Observe that (34) can be viewed as representation formulae for the Riemann invariants in function of their boundary values.

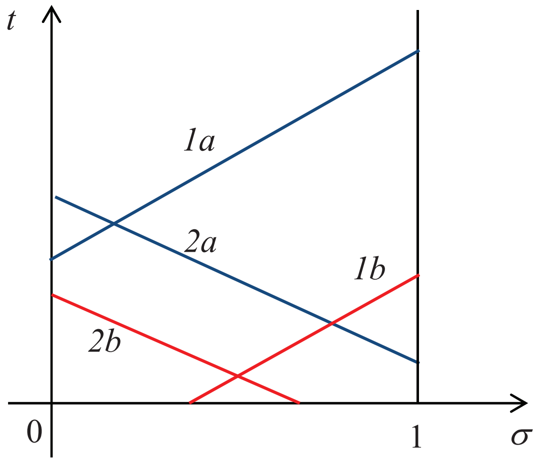

Fig. 3. Forward and backward characteristics

For those characteristics which can be extended to the entire interval $(0,l)$ - see figure 3, lines 1a and 2a - (34) will give

$$\begin{aligned}
r^+(0,t) &= r^+(l, t+l\sqrt{\rho/E}) -\\
&- \sqrt{\frac{\rho}{2E}}\int_0^l f_a(\sigma, t+\sigma\sqrt{\rho/E})d\sigma\\
r^-(l,t) &= r^-(0, t+l\sqrt{\rho/E}) +\\
&+ \frac{a}{2}\sqrt{\frac{\rho}{E}}\int_0^l f_a(\sigma, t+(l-\sigma)\sqrt{\rho/E})d\sigma
\end{aligned} \tag{35}$$

Introduce now the functions

$$\begin{aligned}
&y^+(t) := r^+(l,t)\ ,\ y^-(t) := r^-(0,t)\\
&\eta^\pm(t) := y^\pm(t+l\sqrt{\rho/E})
\end{aligned} \tag{36}$$

to obtain from (35)

$$\begin{aligned}
r^+(0,t) &= \eta^+(t+l\sqrt{\rho/E}) -\\
&- \sqrt{\frac{\rho}{2E}}\int_0^l f_a(\sigma, t+\sigma\sqrt{\rho/E})d\sigma\\
r^-(l,t) &= \eta^-(t+l\sqrt{\rho/E}) +\\
&+ \frac{a}{2}\sqrt{\frac{\rho}{E}}\int_0^l f_a(\sigma, t+(l-\sigma)\sqrt{\rho/E})d\sigma
\end{aligned} \tag{37}$$

We substitute now (36) and (37) in the boundary conditions of (30) to obtain, after certain manipulation

$$
\begin{aligned}
& m_0\ddot{z}_1 + \left(c_0 + \frac{(c_0')^2}{a+c_0'}\right)\dot{z}_1 + k_0 z_1 + \\
& \quad + \frac{c_0'\sqrt{2}}{a+c_0'}\eta^-(t - l\sqrt{\rho/E}) = f_0(t) \\
& m_1\ddot{z}_2 + (a+c_1)\dot{z}_2 + k_1 z_1 - \\
& \quad - a\sqrt{2}\eta^+(t - l\sqrt{\rho/E}) + f_1(\dot{z}_2) = 0 \\
& \eta^+(t) - \frac{a-c_0'}{a(a+c_0')}\eta^-(t - l\sqrt{\rho/E}) = \frac{c_0'\sqrt{2}}{a+c_0'}\dot{z}_1 + \\
& \quad + \sqrt{\frac{\rho}{2E}}\int_0^l f_a(\sigma, t + \sigma\sqrt{\rho/E})\mathrm{d}\sigma \\
& \eta^-(t) + a\eta^+(t - l\sqrt{\rho/E}) = a\sqrt{2}\dot{z}_2 - \\
& \quad - \frac{a}{2}\sqrt{\frac{\rho}{E}}\int_0^l f_a(\sigma, t + (l-\sigma)\sqrt{\rho/E})\mathrm{d}\sigma
\end{aligned}
\tag{38}
$$

The solution of (38) can be constructed by steps on intervals $(kl\sqrt{\rho/E}, (k+1)l\sqrt{\rho/E})$, $k = 0, \pm1, \pm2 \ldots$ The components $\eta^\pm(t)$ will, generally speaking display finite discontinuities at $kl\sqrt{\rho/E}$ if a discontinuity occurs at $t = 0$ between the initial condition $\eta_0^\pm(0-)$ and the solution $\eta^\pm(0+)$. This discontinuity is a result of the mismatch between the initial and the boundary conditions of (26).

On the other hand, the construction by steps of the solution of (38) requires knowledge of the initial conditions. If the initial conditions for $z_i(t)$, $\dot{z}_i(t)$, $i = 1, 2$, clearly migrate from (26) or (12), the initial conditions for $\eta^\pm(t)$ on $(-l\sqrt{\rho/E}, 0)$ have to be constructed starting from the initial conditions of (12) on $(0, l)$. Let

$$
u(x, 0) = u_0(x) \ , \ u_t(x, 0) = u_1(x) \ ; \ 0 \le x \le l
$$

Then we shall have for (26) that

$$
\begin{aligned}
& v(x, 0) = u_t(x, 0) = u_1(x) \\
& w(x, 0) = E(x)\Gamma(x)u_x(x, 0) = E(x)\Gamma(x)u_0'(x)
\end{aligned}
\tag{39}
$$

Also for the Riemann invariants we deduce from (28)

$$
\begin{aligned}
& r^+(x, 0) = \frac{1}{\sqrt{2}}\left[u_1(x) - \frac{1}{a(x)}E(x)\Gamma(x)\Gamma(x)u_0'(x)\right] \\
& r^-(x, 0) = \frac{1}{\sqrt{2}}\left[a(x)u_1(x) + E(x)\Gamma(x)\Gamma(x)u_0'(x)\right]
\end{aligned}
\tag{40}
$$

To obtain the initial conditions for $\eta^\pm(t)$ on $(-l\sqrt{\rho/E}, 0)$ from (40), we shall consider again the Riemann invariants along those characteristics which cannot be extended on the entire interval $(0, l)$ - see the lines 2a, 2b on figure 3; they cross the axis $t = 0$ between $x = 0$ and $x = l$.

Consider therefore $\varphi^\pm(\sigma)$ of (32) satifying (33). The characteristic line $t^+(\sigma; x, t)$ may cross the horizontal (abscissa) line at $\sigma = x - t\sqrt{E/\rho} \in (0, l)$. We integrate the equation of $\varphi^+(\sigma)$ between $\sigma = l$ and $\sigma = x - t\sqrt{E/\rho}$ to obtain first

$$
\begin{aligned}
& \varphi^+(x - t\sqrt{E/\rho}) = r^+(x - t\sqrt{E/\rho}, 0) = \\
& \quad = r^+(l, t + (l-x)\sqrt{\rho/E}) - \\
& \quad - \sqrt{\frac{\rho}{2E}}\int_{x - t\sqrt{E/\rho}}^l f_a(\sigma, t + (\sigma - x)\sqrt{\rho/E})\mathrm{d}\sigma
\end{aligned}
\tag{41}
$$

Using the notations of (36), (41) is re-written as

$$
\begin{aligned}
& \eta^+(t - x\sqrt{\rho/E}) = r^+(x - t\sqrt{E\rho}) + \\
& \quad + \sqrt{\frac{\rho}{2E}}\int_{x - t\sqrt{E/\rho}}^l f_a(\sigma, t + (\sigma - x)\sqrt{\rho/E})\mathrm{d}\sigma
\end{aligned}
\tag{42}
$$

Since $x - t\sqrt{E/\rho} \in (0, l)$, it follows that $t - x\sqrt{\rho/E} \in (-l\sqrt{\rho/E}, 0)$. Denoting $\theta := t - x\sqrt{\rho/E}$, we obtain from (42)

$$
\begin{aligned}
& \eta_0^-(\theta) = r_0^+(-\theta\sqrt{E/\rho}, 0) + \\
& \quad + \sqrt{\frac{\rho}{2E}}\int_{-\theta\sqrt{E/\rho}}^l f_a(\sigma, \theta + \sigma\sqrt{E/\rho})\mathrm{d}\sigma
\end{aligned}
\tag{43}
$$

with $r_0^+(\cdot)$ taken from (40). In the same way we obtain for $\eta_0^-(\theta)$

$$
\begin{aligned}
& \eta_0^-(\theta) = r_0^-(l + \theta\sqrt{E/\rho}, 0) - \\
& \quad - \frac{a}{2}\sqrt{\frac{\rho}{E}}\int_0^{l + \theta\sqrt{E/\rho}} f_a(\sigma, \theta + (l - \sigma)\sqrt{\rho/E})\mathrm{d}\sigma
\end{aligned}
\tag{44}
$$

with $r_0^-(\cdot)$ taken from (40).

Summarizing the development of this Section, we can state the following result

Theorem 1: Consider the system (26) and let $\{z_i(t), v(x, t, w(x, t)\}$ be a classical solution of it, defined by the initial conditions $\{z_i(0), \dot{z}_i(0), u_0(x), u_1(x)\}$ with $u_k(x)$, $k = 0, 1$ defined on $[0, l]$. Let $\eta^\pm(t)$ be defined by (36), with the Riemann invariants $r^\pm(x, t)$ defined by (28). Then $\{z_i(t), \eta^\pm(t)\}$ is a solution of the system of functional differential equations with deviated argument (38) with possible finite discontinuities of $\eta^\pm(t)$ at $kl\sqrt{\rho/E}$, $k = 0, \pm1, \pm2, \ldots$, defined by the initial conditions $\{z_i(0), \dot{z}_i(0), \eta_0^\pm(\theta), -l\sqrt{\rho/E} \le \theta < 0\}$, where $\eta^\pm(\theta)$ are given by (43), (44) and (40).

Conversely, let $\{z_i(t), \eta^\pm(t)\}$ be a solution of (38) with its initial conditions $\{z_i(0), \dot{z}_i(0), \eta_0^\pm(\theta), -l\sqrt{\rho/E} \le \theta < 0\}$, where $\eta_0^\pm(\theta)$ are sufficiently smooth (e.g. $\mathcal{C}^1$). Then

$\{z_i(t), v(x,t,w(x,t)\}$ is a classical solution of (26) with $\{v(x,t), w(x,t)\}$ being defined by (27) and $r^{\pm}(x,t)$ by

$$
\begin{aligned}
r^+(x,t) &= \eta^+(t - x\sqrt{\rho/E}) - \\
&-\sqrt{\frac{\rho}{E}}\int_x^l f_a(\sigma, t + (\sigma - x)\sqrt{\rho/E})\mathrm{d}\sigma \\
r^-(x,t) &= \eta^-(t + (x-l)\sqrt{\rho/E}) + \\
&+\frac{a}{2}\sqrt{\frac{\rho}{E}}\int_0^x f_a(\sigma, t + (x-\sigma)\sqrt{\rho/E})\mathrm{d}\sigma
\end{aligned}
\tag{45}
$$

The initial conditions of this solution result for $t = 0$.

Summarizing, Theorem 1 establishes a one-to-one correspondence between the solutions of (26) (under constant parameters and zero internal damping $c(x) \equiv 0$) and the solutions of (38). As a consequence, any property established for one of the two mathematical objects is automatically projected back on the other. In our paper this concerns asymptotic stability of the zero solution in the autonomous case ($f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$) with constant parameters and $c(x) \equiv 0$. This problem will be tackled in the next section.

VI. The asymptotic stability

In the most general case of (12), in the autonomous case, we showed existence of the 0 equilibrium and its non-asymptotic stability in the sense of the metrics induced by the Lyapunov functional (23) - see the identity (24).

In the case of the constant parameters and with $c(x) \equiv 0$, making use of the variables v, w defined by (25), the Lyapunov functional(23) becomes

$$
\begin{aligned}
&\mathcal{V}^\star(t) = \mathcal{V}(z_1(t), z_2(t), \dot z_1(t), \dot z_2(t), v(\cdot,t), w(\cdot,t)) \equiv \\
&\equiv \frac{1}{2}\Big\{k_0 z_1^2(t) + m_0\dot z_1^2(t) + k_1 z_2^2(t) + m_1 \dot z_1^2(t) + \\
&+\int_0^l [\rho\Gamma v(x,t)^2 + (E\Gamma)^{-1} w(x,t)^2]\mathrm{d}x\Big\}
\end{aligned}
\tag{46}
$$

and is subject to

$$
\frac{\mathrm{d}\mathcal{V}^\star}{\mathrm{d}t} + c_0\dot z_1^2(t) + c_1 \dot z_2^2(t) + f_1(\dot z_2(t))\dot z_2(t) \equiv 0 \tag{47}
$$

hence $\mathcal{V}$ is non-increasing along the solutions of the autonomous system (12) with constant parameters and $c(x) \equiv 0$. Express now $\mathcal{V}$ in the variables of the Riemann invariants using (27) - with $a = \Gamma\sqrt{\rho E}$ being also constant

$$
\begin{aligned}
&\mathcal{V}^\star(t) = \mathcal{V}(z_1(t), z_2(t), \dot z_1(t), \dot z_2(t), r^+(\cdot,t), r^-(\cdot,t)) \equiv \\
&\equiv \frac{1}{2}\Big\{k_0 z_1^2(t) + m_0\dot z_1^2(t) + k_1 z_2^2(t) + m_1 \dot z_1^2(t) + \\
&+\int_0^l [\rho\Gamma r^+(x,t)^2 + (E\Gamma)^{-1} r^-(x,t)^2]\mathrm{d}x\Big\}
\end{aligned}
\tag{48}
$$

which is subject also to (47). It follows that the zero solution of (30) - with $f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$ - is Lyapunov stable in the sense of the metrics induced by the Lyapunov functional (48).

In order to obtain asymptotic stability of the zero equilibrium of (30) - and of (26) under the acting assumptions - we shall apply the Barbashin-Krasovskii-LaSalle invariance principle. The theorem allowing application of the Barbashin-Krasovskii-LaSalle invariance principle is Theorem 9.8.2 of [14], p.293 concerning neutral functional differential equations. Consequently, we shall turn to system (38) which is of neutral type. This system is considered also under the assumption that $f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$ and its only equilibrium will be in this case at the origin.

The Lyapunov functional (48) can be expressed in the language of (38) by making use of the representation formulae (45) with $f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$. It results

$$
\begin{aligned}
&\mathcal{V}^\star(t) = \mathcal{V}(z_1(t), z_2(t), \dot z_1(t), \dot z_2(t), \eta^+(t+\cdot), \eta^-(t+\cdot)) \equiv \\
&\equiv \frac{1}{2}\Big\{k_0 z_1^2(t) + m_0\dot z_1^2(t) + k_1 z_2^2(t) + m_1 \dot z_1^2(t) + \\
&+\int_{-l\sqrt{\rho/E}}^0 [a\eta^+(t+\theta)^2 + (a)^{-1}\eta^-(t+\theta)^2]\mathrm{d}\theta\Big\}
\end{aligned}
\tag{49}
$$

which is also subject to (47) hence $\mathcal{V}$ is non-increasing along the solutions of (38). Therefore the zero solution of (38) results Lyapunov stable in the metrics induced by the Lyapunov functional (49) itself.

In order to obtain asymptotic stability, we shall consider the set where the derivative of $\mathcal{V}$ vanishes: it follows from (47) that

$$
\dot z_1(t) \equiv \dot z_2(t) \equiv 0 \tag{50}
$$

what gives that $z_1(t) \equiv$ const, $z_2(t) \equiv$ const. System (38) restricted to the set defined by (50) will be

$$
\begin{aligned}
&k_0\bar z_1 + \frac{c_0'\sqrt{2}}{a + c_0'}\eta^-(t - l\sqrt{\rho/E}) = 0 \\
&k_1\bar z_2 - a\eta^+(t - l\sqrt{\rho/E}) = 0 \\
&\eta^+(t) - \frac{a - c_0'}{a(a + c_0')}\eta^-(t - l\sqrt{\rho/E}) = 0 \\
&\eta^-(t) + a\eta^+(t - l\sqrt{\rho/E}) = 0
\end{aligned}
\tag{51}
$$

It is not difficult to see that the only solutions of (52) are the constant ones: they thus define the largest invariant set included in the set defined by (50). The constant solutions of (52) are the solutions of the linear homogeneous system

$$
\begin{aligned}
&k_0\bar z_1 + \frac{c_0'\sqrt{2}}{a + c_0'}\bar\eta^- = 0 \\
&k_1\bar z_2 - a\bar\eta^+ = 0 \\
&\bar\eta^+ - \frac{a - c_0'}{a(a + c_0')}\bar\eta^- = 0 \\
&\bar\eta^- + a\bar\eta^+ = 0
\end{aligned}
\tag{52}
$$

Its determinant reads

$$\Delta = \begin{vmatrix} k_0 & 0 & 0 & \frac{c_0'\sqrt{2}}{a+c_0'} \\ 0 & k_1 & -a & 0 \\ 0 & 0 & 1 & -\frac{a-c_0'}{a(a+c_0')} \\ 0 & 0 & a & 1 \end{vmatrix} = \frac{2ak_0k_1}{a+c_0'} \neq 0 \qquad (53)$$

Therefore the only invariant set included in the set defined by (50) is the equilibrium at the origin since (**??**) has only the zero solution because of (53). The Barbashin-Krasovskii-LaSalle invariance principle expressed by Theorem 9.8.2 of [14], p.293 will thus give asymptotic stability provided the difference operator of (38) is asymptotically stable. The difference operator is defined by the matrix

$$D = \begin{pmatrix} 0 & \frac{a-c_0'}{a(a+c_0')} \\ -a & 0 \end{pmatrix} \qquad (54)$$

and it will result asymptotically (even exponentially) stable if D has its eigenvalues inside the unit disk of $\mathbb{C}$ i.e. with their modulus less than 1. But this assertion is true since the characteristic equation of (54) is

$$\lambda^2 - \frac{a-c_0'}{a+c_0'} = 0$$

and $|(a-c_0')/(a+c_0')| < 1$. The zero equilibrium of (38) with $f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$ is thus asymptotically stable. Making use of the representation formulae (45) we obtain

$$\lim_{t\to\infty} r^+(x,t) = \lim_{t\to\infty} r^-(x,t) = 0 \qquad (55)$$

hence the zero equilibrium of (30) with $f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$ is also asymptotically stable. Moreover, using (27) we obtain

$$\lim_{t\to\infty} v(x,t) = \lim_{t\to\infty} w(x,t) = 0 \qquad (56)$$

what gives asymptotic stability of the zero equilibrium of (26) with $f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$ in the case of constant parameters and zero distributed damping $c(x) \equiv 0$. The results can be summarized in

Theorem 2: Consider system (26) with zero distributed damping $c(x) \equiv 0$, constant parameters ρ, Γ, E and with the external forces $f_0(t) \equiv 0$, $f_a(x,t) \equiv 0$. Its equilibrium at 0 is globally asymptotically stable and this property holds for systems (30) and (38) also.

VII. Conclusions and perspective

We have deduced in this paper the dynamics of a quite simple structure in Contact Mechanics, describing vibroimpact devices at the basic level. The variational principle of Hamilton for systems with distributed parameters was applied. The contact was modeled by a rather simple version of the complementarity Hertz-Signorini-Moreau condition leading to a sector restricted nonlinearity. Based on the energy identity, an energy-like Lyapunov functional has been associated to the dynamical system viewed without external forces in order to assess stability of the equilibrium at the origin. Being a "weak" Lyapunov functional, the energy functional ensured only the non-asymptotic stability in the metrics induced by the Lyapunov functional itself. For the asymptotic stability we used the Barbashin-Krasovskii-LaSalle invariance principle applied to an associated system of neutral functional differential equations whose solutions are in one-to-one correspondence with the solutions of the system under analysis. In this way asymptotic stability was obtained. However application of the Barbashin-Krasovskii-LaSalle invariance principle was possible due to asymptotic stability of the difference operator associated to the system of neutral type. At its turn, this stability property was obtained by taking into account the elastic strain in the mechanical system, induced by the acting external force at the boundary.

The research reported in this paper can be extended in several directions: we mention but more complicated Hertz-Signorini-Moreau complementarity conditions the and the analysis of the system viewed as a system with impulses - due to the non-penetrating contact. Dynamic steady states (non-equilibrium) induced by external forces can be also explored, leading to such mathematical problems as Levinson dissipativeness, forced periodic or almost periodic oscillations and other.

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