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On $\mathfrak{S}$-entropy Analysis of Linear Stochastic Systems in State Space

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***Abstract*—In this paper the problem of random disturbance attenuation capabilities in linear continuous systems is studied. It is supposed that the system operates under random disturbances with bounded σ-entropy level. $\mathfrak{S}$-entropy norm indicates a performance index of the continuous system on the set of the random signals with bounded σ-entropy. This paper presents a time-domain solution to the calculation of σ-entropy norm of the continuous linear time-invariant system. $\mathfrak{S}$-entropy norm $|\!|\!|F|\!|\!|_{\mathfrak{S}}$ is defined after solving coupled matrix equations: one algebraic Riccati equation, one nonlinear equation over log determinant function, and two Lyapunov equations.**

***Index Terms*—Stochastic systems, linear systems, system sensitivity, entropy function.**

I. Introduction

The solution of the control problems in linear control systems should guarantee the internal stability of the closed-loop system, provide desired robustness and satisfy additional performance criteria. The external disturbance rejection capability is one of the most popular performance criteria in the linear control. In order to solve the disturbance rejection problem, the designer should define the set of the perturbing signals as well as a measure of its attenuation. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms have become the most popular cost measures in the feedback optimization. Both norms have physical meaning with respect to the perturbing signals. Thus, $\mathcal{H}_2$ norm indicates output dispersion of the controlled output in presence of white Gaussian noise while $\mathcal{H}_\infty$ norm of the system stands for the maximum error energy gain for disturbance input with bounded energy. However, both performance criteria have substantial drawbacks. An application of $\mathcal{H}_2$ control is quite limited because of in practice external disturbances are noisy signals with the unknown covariance matrix which can change during the time. Additionally, systems closed by $\mathcal{H}_2$ controllers lack of robustness. Alternatively, $\mathcal{H}_\infty$ controllers may lead to excessive energy consumptions if external disturbances are slightly correlated noises. This facts put researchers onto an idea to find compromises between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimization approaches.

In the late 1980 a so-called entropy $\mathcal{H}_\infty$ control is appeared. There are a lot of papers devoted to the entropy $\mathcal{H}_\infty$ control for continuous and discrete-time systems. The key idea of the minimum entropy $\mathcal{H}_\infty$ control approach is to find a solution to the LQG control problem with an additional constraint on the system's entropy. The entropy function, suggested in [1] is an adaptation of the method of Arov and Krein [2]. Minimum entropy $\mathcal{H}_\infty$ control theory have become a simple tradeoff between the (upper bounds on the) $\mathcal{H}_\infty$ and LQG objectives [3]. The $\mathcal{H}_\infty$ objectives reflect both the robust stability and performance requirements, where an noise is taken to be of bounded energy. The tradeoff is against the LQG measure of performance where the noise is taken to be Gaussian and white. One can refer to [1], [4]–[6] for more details.

A problem of minimax LQG control, solved in [7] involves the relative entropy function to describe possible uncertainties in a plant model. The idea of minimax LQG control is to find controller that minimizes the linear quadratic functional with respect to the worst uncertainties in entropy sense.

In contrast with above mentioned theoretic-information approaches, the anisotropy-based control theory applies the relative entropy constraint as a set of random colored signals with unknown covariance. The key term of the anisotropy-based control theory is a mean anisotropy of the random signal. The mean anisotropy is a scalar nonnegative parameter. It has physical meaning as distance measure between white Gaussian noise and the random signal itself. The mean anisotropy also defines a set of the random input disturbances which "distance" from white Gaussian noise in terms of the relative entropy does not exceed a given value. Similar to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, the anisotropic norm of the linear system defines a performance index of system on the set of the random signals with bounded mean anisotropy level. In the anisotropy-based theory a tradeoff between LQG/$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control is reached using the mean anisotropy of the input signal as a restrictive set. In should be noted that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are particular cases of the anisotropic norm. A fairly good survey on this theory is given in [8].

Unfortunately, the anisotropy-based control theory is limited by discrete-time systems. A further extension idea of the anisotropy-based control theory to continuous linear systems has lead to the appearance of σ-entropy theory. This theory has an another axiomatics that allows to get rid of the relative entropy and extend a set of the signal under consideration. The main difference between these theories is that σ-entropy does not involve any reference signal. Similar to the mean anisotropy, σ-entropy is a scalar parameter which defines the

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set of input signals in terms of spectral density using the entropy integral. The σ-entropy norm indicates a performance index of continuous system on the set of the random signals with bounded σ-entropy. Similar to the anisotropic norm, the σ-entropy norm lies between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the system. The development of the analysis and control methods using σ-entropy constraint seems attractive since it generalizes all above mentioned approaches within unified framework.

The problem of the σ-entropy analysis of continuous-time linear systems in the frequency domain has been solved in [9]. In paper [10] a solution to the σ-entropy analysis in time domain was obtained for strictly proper transfer functions (i.e. with $D = 0$). Current research presents the solution of the σ-entropy analysis of continuous-time linear systems in time domain in general form, i.e. with $D \neq 0$. This solution has lead to a different set of equations.

The rest of the paper is organized as follows. In Section II necessary background and basic definitions of σ-entropy theory are given. Section III presents the main result of the paper. Conclusions and future plans are highlighted in Section IV.

II. Preliminary Results and Problem Statement

We deal with the linear time-invariant system with the state-space realization given by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \qquad x(0) = 0, \\ z(t) = Cx(t) + Dw(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a system state, $z(t) \in \mathbb{R}^p$ is an output signal, $w(t) \in \mathbb{R}^m$ is an input disturbance. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ are constant real matrices. The matrix A is assumed to be Hurwitz and the stochastic signal $w(t)$ is also assumed to satisfy the following conditions:

$$\mathbf{E}\left[w(t)\right] = 0,$$

either the $\mathcal{L}_2$ norm

$$\left\|w(t)\right\|_2 = \sqrt{\int_{-\infty}^{+\infty} \mathbf{E}\Big[\left|w(t)\right|^2\Big]\, dt} \tag{2}$$

or the power norm of the random signal

$$\left\|w(t)\right\|_{\mathcal{P}} = \sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{E}\Big[\left|w(t)\right|^2\Big]\, dt} \tag{3}$$

is finite. Here $\mathbf{E}[\cdot]$ is an expected value of $(\cdot)$ and $|\cdot|$ is the Euclidean norm of a vector.

In accordance with [9], let us generalize the description of the properties (2 – 3) and define the $\mathfrak{N}$ norm of the random signal $w(t)$ as follows:

$$\left\|w(t)\right\|_{\mathfrak{N}}^2 = \mathfrak{N}\big(w^{\mathrm{T}}(t)\, w(t)\big).$$

In this definition $\mathfrak{N}$ is the linear operator, which transforms the Euclidean norm $\left|w(t)\right|^2 = w(t)^{\mathrm{T}} w(t)$ into the $\mathcal{L}_2$ or power norm of the stochastic signal $w(t)$ according to following rule:

$$\mathfrak{N}(\cdot) = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \mathbf{E}[\cdot]\, dt & \text{if } \left\|w(t)\right\|_2 < \infty \\ \displaystyle\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{E}[\cdot]\, dt & \text{if } \left\|w(t)\right\|_{\mathcal{P}} < \infty. \end{cases}$$

Now we can define the correlation convolution $K_w(\tau)$ of the signal $w(t)$ as in [9], [11]

$$K_w(\tau) = \mathfrak{N}\big(w(t+\tau)\, w^{\mathrm{T}}(t)\big)$$

and the spectral density $S_w(\omega)$ of $w(t)$ by using the Fourier transform of $K_w(\tau)$:

$$S_w(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K_w(\tau)\, \mathrm{e}^{-i\omega\tau}\, d\tau.$$

The correlation convolution $K_w(\tau)$ can be obtained from $S_w(\omega)$ by the inverse Fourier transform as follows:

$$K_w(\tau) = \int_{-\infty}^{+\infty} S_w(\omega)\, \mathrm{e}^{i\omega\tau} d\omega.$$

Then the $\mathfrak{N}$ norm of $w(t)$ is equal to:

$$\left\|w(t)\right\|_{\mathfrak{N}}^2 = \int_{-\infty}^{+\infty} \operatorname{tr} S_w(\omega) d\omega.$$

Similarly, the $\mathfrak{N}$ norm of an output signal $z(t)$ is given by:

$$\left\|z(t)\right\|_{\mathfrak{N}}^2 = \int_{-\infty}^{+\infty} \operatorname{tr} S_z(\omega)\, d\omega,$$

where $S_z(\omega)$ is the spectral density of $z(t)$. For the system (1) with the transfer matrix $F(s) = C(sI - A)^{-1}B + D$ the spectral density $S_z(\omega)$ is equal to [12]:

$$S_z(\omega) = F(i\omega)\, S_w(\omega)\, F^*(i\omega),$$

here F^* is the complex conjugate transpose of a matrix F.

For the system (1) with the input signal (2) or (3), define the system gain Θ as a ratio of $\mathfrak{N}$ norm of the system output $z(t)$ to $\mathfrak{N}$ norm of the input signal $w(t)$:

$$\Theta = \frac{\left\|z(t)\right\|_{\mathfrak{N}}}{\left\|w(t)\right\|_{\mathfrak{N}}} = \frac{\sqrt{\displaystyle\int_{-\infty}^{+\infty} \operatorname{tr}\big[\Lambda(\omega) S_w(\omega)\big]\, d\omega}}{\sqrt{\displaystyle\int_{-\infty}^{+\infty} \operatorname{tr} S_w(\omega)\, d\omega}},$$

where $\Lambda(\omega) = F^*(i\omega) F(i\omega)$.

Definition 1: Let $w(t)$ be a stochastic input signal with the finite $\mathfrak{N}$ norm, then the σ-entropy $\mathfrak{S}(S_w)$ of the signal $w(t)$ with the spectral density S_w is defined as:

$$\mathfrak{S}(S_w) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(\omega) \ln\det \frac{mS_w(\omega)}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \operatorname{tr} S_w(\omega')\, d\omega'}\, d\omega.$$

Here $\varphi(\omega) = \dfrac{\omega_0^2}{\omega_0^2+\omega^2}$ is the function which provides the convergence of the integral

$$\int_{-\infty}^{+\infty} \ln\det S_w(\omega)\, d\omega.$$

Remark 1: Scalar parameter ω_0 is a predefined parameter which is connected with system bandwidth. In practical applications ω_0 can be chosen 5-10 times larger than system bandwidth.

Using the above notations we can define the σ-entropy norm of the system (1).

Definition 2: Let $\sigma \geqslant 0$ and let $w(t)$ be a stochastic input disturbance with bounded σ-entropy $\mathfrak{S}(S_w)$. Then the σ-entropy norm $|||F|||_{\mathfrak{S}}$ of the system (1) is defined by:

$$|||F|||_{\mathfrak{S}}^2 = \sup_{\mathfrak{S}(S_w)\leqslant\sigma} \Theta^2 = \sup_{\mathfrak{S}(S_w)\leqslant\sigma} \frac{\int_{-\infty}^{+\infty} \operatorname{tr}\big[\Lambda(\omega) S_w(\omega)\big]\, d\omega}{\int_{-\infty}^{+\infty} \operatorname{tr} S_w(\omega)\, d\omega}. \tag{4}$$

Based on this definition of the σ-entropy norm, the following frequency domain formulae were established in [9].

Theorem 1: [9] Let F be a system with a state-space realization (1) and let $w(t)$ be a stochastic input disturbance with bounded σ-entropy. Then for any $\sigma \geqslant 0$ the σ-entropy norm (4) is calculated as:

$$|||F|||_{\mathfrak{S}}^2 = \frac{\int_{-\infty}^{+\infty} \varphi(\omega) \operatorname{tr}\big[\Lambda(\omega)\, S_\star(\omega)\big]\, d\omega}{\int_{-\infty}^{+\infty} \varphi(\omega) \operatorname{tr} S_\star(\omega)\, d\omega}, \tag{5}$$

where

$$S_\star(\omega) = \big[I - q\Lambda(\omega)\big]^{-1}$$

is a worst case spectral density of the input disturbance and $q \in \big[0, \|F\|_\infty^{-2}\big)$ is the unique solution of the equation:

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(\omega) \ln\det \frac{m\varphi(\omega) S_\star(\omega)}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(\omega') \operatorname{tr} S_\star(\omega')\, d\omega'}\, d\omega = \sigma. \tag{6}$$

Theorem 1 presents the frequency domain technique for calculating the σ-entropy norm. Unfortunately, this approach does not allow to implement numerical tools for the σ-entropy analysis of linear systems. Therefore, the problem of the σ-entropy analysis in time domain has been studied. This problem is formulated as

Problem 1: For a given system F with the state-space realization (1) and known σ-entropy level $\sigma \geqslant 0$ the problem is to find formulas for calculating the σ-entropy norm $|||F|||_{\mathfrak{S}}$ in the state space.

For the sake of convenience we will use the following notation

$$F = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

for the transfer function $F(s) = C(sI-A)^{-1}B + D$ of the system (1).

Definition 3: A system with the transfer function $\Upsilon(s)$ is called the inner (or all-pass) system, if:

$$\Upsilon^*(i\omega)\Upsilon(i\omega) = I.$$

Lemma 1: [13] Suppose $\Upsilon = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ and X is the observability Gramian. Then Υ is inner if and only if:

$$\begin{aligned} A^{\mathrm{T}}X + XA + C^{\mathrm{T}}C &= 0, \\ D^{\mathrm{T}}C + B^{\mathrm{T}}X &= 0, \\ D^{\mathrm{T}}D &= I. \end{aligned}$$

III. Main Result

Before we formulate the main result of the paper, several intermediate results should be highlighted. These results are formulated in lemmas.

Lemma 2: Consider a system G with the state-space realization $G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right]$. Then

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\omega_0^2}{\omega_0^2+\omega^2} \operatorname{tr}\big[G^*(i\omega)\, G(i\omega)\big]\, d\omega = = \operatorname{tr}\left\{\begin{bmatrix} B_G \\ D_G \end{bmatrix}^{\mathrm{T}} \mathit{\Gamma} \begin{bmatrix} B_G \\ D_G \end{bmatrix}\right\}, \tag{7}$$

where $\mathit{\Gamma}$ is the unique solution of the Lyapunov equation:

$$\begin{bmatrix} A_G & 0 \\ C_G & -\omega_0 I \end{bmatrix}^{\mathrm{T}} \mathit{\Gamma} + \mathit{\Gamma} \begin{bmatrix} A_G & 0 \\ C_G & -\omega_0 I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \omega_0^2 I \end{bmatrix} = 0. \tag{8}$$

Proof: Consider the following system Ω:

$$\Omega = \left[\begin{array}{c|c} -\omega_0 I & I \\ \hline \omega_0 I & 0 \end{array}\right] = \frac{\omega_0}{\omega_0 + s} I. \tag{9}$$

Then the integral in the left side of (7) can be rewritten as:

$$\int_{-\infty}^{+\infty} \mathrm{tr}\left\{\left[\frac{\omega_0}{\omega_0+i\omega}\,G(i\omega)\right]^*\left[\frac{\omega_0}{\omega_0+i\omega}\,G(i\omega)\right]\right\}d\omega = \\ = \int_{-\infty}^{+\infty}\mathrm{tr}\big[H^*(i\omega)\,H(i\omega)\big]\,d\omega, \quad (10)$$

where

$$H(s) = \Omega(s)G(s) = \frac{\omega_0}{\omega_0+s}\Big[D_G + C_G\big(sI - A_G\big)^{-1}B_G\Big].$$

It is easy to see that:

$$H = \Omega G = \left[\begin{array}{c|c} -\omega_0 I & I \\ \hline \omega_0 I & 0\end{array}\right]\left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G\end{array}\right] = \\ = \left[\begin{array}{cc|c} A_G & 0 & B_G \\ C_G & -\omega_0 I & D_G \\ \hline 0 & \omega_0 I & 0\end{array}\right]. \quad (11)$$

The integral in the right hand side of (10) is the scaled square of the $\mathcal{H}_2$ norm of the system H, i.e.

$$\int_{-\infty}^{+\infty}\mathrm{tr}\big[H^*(i\omega)\,H(i\omega)\big]\,d\omega = 2\pi\big\|H\big\|_2^2.$$

In accordance with (11):

$$H = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H\end{array}\right] = \left[\begin{array}{cc|c} A_G & 0 & B_G \\ C_G & -\omega_0 I & D_G \\ \hline 0 & \omega_0 I & 0\end{array}\right].$$

Noting that $D_H = 0$, then the $\mathcal{H}_2$ norm of the system H is equal to [14]:

$$\big\|H\big\|_2^2 = \mathrm{tr}\big(B_H^{\mathrm{T}}\varGamma B_H\big),$$

where $\varGamma$ is the unique solution of the Lyapunov equation:

$$A_H^{\mathrm{T}}\varGamma + \varGamma A_H + C_H^{\mathrm{T}}C_H = 0.$$

The explicit notation of these equations gives us:

$$\big\|H\big\|_2^2 = \mathrm{tr}\left\{\begin{bmatrix}B_G\\D_G\end{bmatrix}^{\mathrm{T}}\varGamma\begin{bmatrix}B_G\\D_G\end{bmatrix}\right\},$$

$$\begin{bmatrix}A_G & 0\\ C_G & -\omega_0 I\end{bmatrix}^{\mathrm{T}}\varGamma + \varGamma\begin{bmatrix}A_G & 0\\ C_G & -\omega_0 I\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \omega_0^2 I\end{bmatrix} = 0,$$

which coincide with (7 – 8). ■

Lemma 3: Suppose $G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G\end{array}\right]$ is a square $m\times m$ transfer function matrix. Then

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega_0^2}{\omega_0^2+\omega^2}\ln\det\left[\frac{\omega_0^2}{\omega_0^2+\omega^2}\,G^*(i\omega)G(i\omega)\right]d\omega = \\ = \omega_0\ln\det\Big[D_G + C_G\big(\omega_0 I - A_G\big)^{-1}B_G\Big] - m\omega_0\ln 2. \quad (12)$$

Proof: Recall that [1]:

$$\ln\det\big[H^*(i\omega)H(i\omega)\big] = 2\ln\det\big|H(i\omega)\big|.$$

In accordance with (9) the left hand side of (12) takes form:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega_0^2}{\omega_0^2+\omega^2}\ln\det\left\{\Big[\Omega(i\omega)G(i\omega)\Big]^*\Omega(i\omega)G(i\omega)\right\}d\omega = \\ = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\omega_0^2}{\omega_0^2+\omega^2}\ln\det\big|H(i\omega)\big|\,d\omega.$$

Using the fact that transfer matrix $H(i\omega)$ with the state-space realization (11) is strictly proper, i.e. $H\in\mathcal{RH}_\infty$, we can apply Poisson integral theorem [15] and get:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\omega_0^2}{\omega_0^2+\omega^2}\ln\det\big|H(i\omega)\big|\,d\omega = \omega_0\ln\det H(\omega_0).$$

One can check that:

$$H(\omega_0) = \frac{1}{2}\Big[D_G + C_G\big(\omega_0 I - A_G\big)^{-1}B_G\Big].$$

Hence,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega_0^2}{\omega_0^2+\omega^2}\ln\det\left[\frac{\omega_0^2}{\omega_0^2+\omega^2}\,G^*(i\omega)G(i\omega)\right]d\omega = \\ = \omega_0\ln\det\Big[D_G + C_G\big(\omega_0 I - A_G\big)^{-1}B_G\Big] - m\omega_0\ln 2.$$

This completes the proof. ■

Theorem 2: Let F be a system with a state-space realization (1) and let $w(t)$ be a stochastic input disturbance with bounded σ-entropy. Then the σ-entropy norm (4) is equal to:

$$|\!|\!|F|\!|\!|_{\mathfrak{S}}^2 = \frac{\mathrm{tr}\left\{M\begin{bmatrix}B\\B\\D\end{bmatrix}^{\mathrm{T}}P\begin{bmatrix}B\\B\\D\end{bmatrix}\right\}}{\mathrm{tr}\left\{M\begin{bmatrix}B\\I\end{bmatrix}^{\mathrm{T}}Q\begin{bmatrix}B\\I\end{bmatrix}\right\}}, \quad (13)$$

where the scalar $q\in\big[0,\|F\|_\infty^{-2}\big)$ and matrices $P>0$, $Q>0$, $R>0$ are the unique solution of the following system of equations (14) – (19):

$$A^{\mathrm{T}}R + RA + qC^{\mathrm{T}}C + L^{\mathrm{T}}M^{-1}L = 0, \quad (14)$$

$$M\big(B^{\mathrm{T}}R + qD^{\mathrm{T}}C\big) = L, \quad (15)$$

$$\big(I - qD^{\mathrm{T}}D\big)^{-1} = M, \quad (16)$$

$$-\frac{\omega_0}{2}\ln\det\frac{mM}{4\,\mathrm{tr}\left\{M\begin{bmatrix}B\\I\end{bmatrix}^{\mathrm{T}}Q\begin{bmatrix}B\\I\end{bmatrix}\right\}} - \\ -\omega_0\ln\det\left\{I + L\Big[\omega_0 I - A - BL\Big]^{-1}B\right\} = s, \quad (17)$$

$$\begin{bmatrix} A+BL & 0 \\ L & -\omega_0 I \end{bmatrix}^{\mathrm{T}} Q + Q \begin{bmatrix} A+BL & 0 \\ L & -\omega_0 I \end{bmatrix} + \\ + \begin{bmatrix} 0 & 0 \\ 0 & \omega_0^2 I \end{bmatrix} = 0, \tag{18}$$

$$\begin{bmatrix} A+BL & 0 & 0 \\ BL & A & 0 \\ DL & C & -\omega_0 I \end{bmatrix}^{\mathrm{T}} P + P \begin{bmatrix} A+BL & 0 & 0 \\ BL & A & 0 \\ DL & C & -\omega_0 I \end{bmatrix} + \\ + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_0^2 I \end{bmatrix} = 0. \tag{19}$$

Proof: According to theorem 1, the worst case spectral density of the input is equal to:

$$S_\star(\omega) = \left[I - qF^*(i\omega)F(i\omega)\right]^{-1}.$$

The condition $q \in \left[0,\, \|F\|_\infty^{-2}\right)$ implies that:

$$I - qF^*(i\omega)F(i\omega) > 0, \quad \forall \omega \in \mathbb{R}^1.$$

This guarantees the existence of a spectral factor G such that:

$$\left[I - qF^*(i\omega)F(i\omega)\right]^{-1} = G(i\omega)G^*(i\omega).$$

It is easily verified that the last equality is equivalent to:

$$\begin{bmatrix} \sqrt{q}F^*(i\omega) & G^{-*}(i\omega) \end{bmatrix} \begin{bmatrix} \sqrt{q}F(i\omega) \\ G^{-1}(i\omega) \end{bmatrix} = I.$$

Using the notation $\Upsilon(i\omega) = \begin{bmatrix} \sqrt{q}F(i\omega) \\ G^{-1}(i\omega) \end{bmatrix}$ one can check that $\Upsilon(i\omega)$ is an inner function with the state-space realization [16]:

$$\Upsilon = \left[\begin{array}{c|c} A & B \\ \hline \sqrt{q}C & 0 \\ -M^{-1/2}L & M^{-1/2} \end{array}\right].$$

Applying lemma 1 to the system Υ we get:

$$A^{\mathrm{T}}R + RA + qC^{\mathrm{T}}C + L^{\mathrm{T}}M^{-1}L = 0, \tag{20}$$
$$qD^{\mathrm{T}}C - M^{-1}L + B^{\mathrm{T}}R = 0, \tag{21}$$
$$qD^{\mathrm{T}}D + M^{-1} = I. \tag{22}$$

It follows from (21) and (22) that:

$$M = \left(I - qD^{\mathrm{T}}D\right)^{-1}, \\ L = M\left(B^{\mathrm{T}}R + qD^{\mathrm{T}}C\right)$$

and if we combine this two equations with (20) we obtain the subsystem (14) – (16).

We now consider the σ-entropy constraint (6). Taking into account that $S_\star(\omega) = G(i\omega)G^*(i\omega)$ the integral in the left hand side of (6) may be transformed into:

$$\int_{-\infty}^{+\infty} \varphi(\omega) \ln\det \frac{m\varphi(\omega)G^*(i\omega)G(i\omega)}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(\omega')\,\mathrm{tr}\left[G^*(i\omega')G(i\omega')\right]d\omega'}\, d\omega =$$

$$= \int_{-\infty}^{+\infty} \frac{\omega_0^2}{\omega_0^2+\omega^2} \ln\det\left[\frac{\omega_0^2}{\omega_0^2+\omega^2}\, G^*(i\omega)G(i\omega)\right] d\omega - \tag{23}$$

$$- m\int_{-\infty}^{+\infty} \frac{\omega_0^2}{\omega_0^2+\omega^2}\, d\omega \times$$

$$\times \ln\left\{\frac{1}{2\pi m} \int_{-\infty}^{+\infty} \frac{\omega_0^2}{\omega_0^2+(\omega')^2}\,\mathrm{tr}\left[G^*(i\omega')G(i\omega')\right]d\omega'\right\}. \tag{24}$$

Applying lemma 3 to the addend (23), lemma 2 to the multiplier (24) and noting that:

$$\int_{-\infty}^{+\infty} \frac{\omega_0^2}{\omega_0^2+\omega^2}\, d\omega = \pi\omega_0,$$

we get [16] the σ-entropy constraint (17) in the state space.

Finally, since the transfer matrix $G(i\omega)$ is a factorization of $S_\star(\omega) = [I - q\Lambda(i\omega)]^{-1}$ in the form $S_\star(\omega) = G(i\omega)G^*(i\omega)$, then (5) can be rewritten as:

$$|||F|||_{\mathfrak{S}}^2 = \frac{\int_{-\infty}^{+\infty} \frac{\omega_0^2}{\omega_0^2+\omega^2}\,\mathrm{tr}\left\{\left[F(i\omega)G(i\omega)\right]^*\left[F(i\omega)G(i\omega)\right]\right\}d\omega}{\int_{-\infty}^{+\infty} \frac{\omega_0^2}{\omega_0^2+\omega^2}\,\mathrm{tr}\left[G^*(i\omega)G(i\omega)\right]d\omega}.$$

Taking into account that:

$$FG = \left[\begin{array}{cc|c} A_G & 0 & B_G \\ BC_G & A & BD_G \\ \hline DC_G & C & DD_G \end{array}\right]$$

and applying lemma 2 to the last expression, we obtain [16]:

$$|||F|||_{\mathfrak{S}}^2 = \frac{\mathrm{tr}\left\{\begin{bmatrix} B_G \\ BD_G \\ DD_G \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} B_G \\ BD_G \\ DD_G \end{bmatrix}\right\}}{\mathrm{tr}\left\{\begin{bmatrix} B_G \\ D_G \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} B_G \\ D_G \end{bmatrix}\right\}} = \\ = \frac{\mathrm{tr}\left\{M\begin{bmatrix} B \\ B \\ D \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} B \\ B \\ D \end{bmatrix}\right\}}{\mathrm{tr}\left\{M\begin{bmatrix} B \\ I \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} B \\ I \end{bmatrix}\right\}},$$

which coincides with (13). According to lemma 2 matrices $Q > 0$ and $P > 0$ are solutions of the equations (18) and (19), respectively. This completes the proof. ■

IV. Conclusions

In this paper we have solved a problem of the σ-entropy analysis of the continuous time linear time invariant system in the time domain. It has been shown that σ-entropy norm $|||F|||_s$ is defined after solving coupled matrix equations: one algebraic Riccati equation, one nonlinear equation over log determinant function, and two Lyapunov equations. In future work we will develop the numerical procedure and numerical tools for σ-entropy computation of stochastic signal $w(t)$ and the σ-entropy norm of linear system.

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